

## Lagrange Multipliers and Economics

1. In class we considered how to optimize utility  $U(x, y)$  with budget constraint  $c_x x + c_y y = B$ . ( $x$  is the number of units of good 1,  $y$  the number of units of good 2,  $c_x$  the price of one unit of good 1 and  $c_y$  the price of one unit of good 2. Let's suppose you are buying the goods, but you could also suppose you are producing them and the  $c$ 's are production costs.) We found that at optimum,  $U_x/c_x = U_y/c_y = \lambda$ .

Let's get concrete. Suppose the utility of having  $x$  guns and  $y$  units of butter is  $U(x, y) = xy$ . (The unit of butter might be a hundred pounds.) Suppose a gun costs \$100 and a unit of butter costs \$50. Suppose the budget constraint is \$400. Suppose currently the economy has 3 guns and 2 units of butter (this does meet the constraint).

- a) Which is currently bigger,  $U_x/c_x$  or  $U_y/c_y$ ?
  - b) Show that with the same budget the society could have more utility. Give a specific allocation choice that is better than (3,2); fractional purchases, like 2.5 guns, is allowed. Did you increase the item with the higher marginal-utility-to-price ratio, or the one with the lower ratio?
  - c) Find the optimal allocation given the budget.
  - d) Suppose society gets \$100 more from somewhere. Give two new allocation plans you could move to from the optimal answer in b). They should result is the same or almost the same utility. Why?
2. Let's generalize the model of the previous problem. It might be that the price of goods varies; e.g., the price per item may go down as you buy more. So let us generalize the constraint to  $C(x) + D(y) = B$ , where  $C(x)$  is the cost to buy  $x$  units of good 1, and similarly for  $D(y)$ . The utility function remains unchanged.  $C(x)$  and  $D(y)$  may be rather arbitrary functions (except they must be differentiable).

- a) Use Lagrange multipliers to show that the optimality condition is now

$$U_x/C'(x) = U_y/D'(y) = \lambda.$$

(**Optimality condition** is the name for the equations you get from the Lagrange approach when you set the partials from the original variables equal to 0.)

- b) What is the interpretation of the ratio  $U_x/C'(x)$ ? of  $U_y/D'(y)$ ?
3. It might be that the cost of the "bundle"  $(x, y)$  is not the sum of two separate functions  $C(x)$  and  $D(x)$  as in problem 2. It might be even more general: a single function  $C(x, y)$ . E.g., as you buy more of good 1, the price of good 2 might go down to be competitive. Find the optimality condition now.
  4. Another optimization situation in basic economics is maximizing production subject to a budget for the costs of capital and labor. Suppose  $P(K, L)$  is how much you can produce if you have  $K$  units of capital and  $L$  units of Labor. Suppose capital costs  $r$  per unit (economists call this cost rent, hence the  $r$ ). Suppose labor costs  $w$  per unit ( $w = \text{wage}$ ). Then you want to maximize  $P(K, L)$  subject to  $rK + wL = B$ .

- a) Find the optimality condition using Lagrange multipliers. (Note:  $r$  and  $w$  are assumed constant.)
- b) What is the interpretation of  $\lambda$ ? Answer this two ways:
- First, assume that the units of  $P(K, L)$  is something concrete. If you are an auto manufacturer, the units would be cars. In order to interpret  $\lambda$ , start by determining the units of  $\lambda$ . (cars per something? something per cars?).
  - Now assume that production is measured in dollars (this is usually what economists do, so they can combine different sorts of production). Now what are the units of  $\lambda$ ? The interpretation of  $\lambda$  is now a certain sort of (pure) multiplier. Explain.
5. Sometimes companies have contracts for a fixed amount of product. Then instead of maximizing production subject to fixed costs, you want to minimize cost for the given production, say,  $P_0$ . Assume you are producing cars. Use the expressions  $P(K, L)$  and  $rK + wL$  as before, with  $r, w$  again constants.
- Find the optimality condition.
  - Interpret  $\lambda$ , when production is in cars. A  $\lambda$  with this sort of interpretation is called a “shadow price”. If another supplier offered to sell you some similar cars (so you wouldn’t have to produce them yourself), at a price lower than  $\lambda$ , what would you do?
6. A motor company makes one type of car and one type of truck. Its revenue is  $R(c, t)$ , where  $c$  is the number of cars it produces (per year) and  $t$  is the number of trucks. Suppose production is constrained by the amount of steel available, where each car needs  $s_c$  units of steel, each truck needs  $s_t$  units of steel, and  $S$  units of steel are available.
- Write the constraint as an equation.
  - Use Lagrange multipliers to determine an optimality equality. What is the interpretation of  $\lambda$  at the optimum?
7. Another motor company makes cars, trucks and vans. Its revenue is  $R(c, t, v)$ , where  $c, t, v$  are the number of cars, trucks, vans (respectively) it produces per year. Suppose production is constrained by the amount of steel available, and by the amount of aluminum available. Assume each car, truck, van needs  $s_c, s_t, s_v$  units of steel, respectively, and  $a_c, a_t, a_v$  units of aluminum, respectively. Suppose  $S$  units of steel and  $A$  of aluminum are available.
- Write the constraints as equations.
  - Use Lagrange multipliers to determine optimality conditions. What are the interpretations of the two  $\lambda$ ’s? Hint: they are both shadow prices.