**Definition 1.** Throughout $f, g, k$ etc will be functions of $x$ (or sometimes $h$). We write $f = o(k)$, or $f(x) = o(k(x))$, and say “$f$ is little oh of $k$”, if

\[ \forall \epsilon > 0 \exists \delta > 0 \ni \left[ |k(x)| \leq \delta \implies |f(x)| \leq \epsilon |k(x)| \right]. \tag{1} \]

Furthermore, we write $f = g + o(k)$ iff $f(x) - g(x) = o(k(x))$.

**Note 1:** The use of equal signs in this definition is an abuse of notation, but a common one (so it must be useful). The symbol $\ni$ here means “such that”. Above we have been careful to say that $\epsilon, \delta > 0$, but henceforth we just refer to $\epsilon$ and $\delta$ without explicitly saying they are positive.

**Note 2:** The most common function $k$ is the identity $k(x) = x$. In this case we write $f(x) = o(x)$ and say that $f$ is little oh of $x$, or just “$f$ is little oh”.

**Note 3:** Intuitively $f = o(k)$ means that $\lim_{x \to 0} \frac{|f(x)|}{|k(x)|} = 0$, but we can’t say it that way because $k(x)$ may equal 0, even for all values of $x$.

**Definition 2.** We write $f = O(k)$, or $f(x) = O(k(x))$, and say $f$ is big Oh of $k$, if

\[ \exists M, \delta \ni \left[ |k(x)| < \delta \implies |f(x)| < M |k(x)| \right]. \tag{2} \]

We write $f = g + O(k)$ iff $f(x) - g(x) = O(k(x))$.

Intuitively $f = O(k)$ means that $|f(x)|/|k(x)|$ is bounded for $k(x)$ small, but we can’t say it that way because $k(x)$ may be 0.

There are even more general definitions of oh and Oh. See the final remark.

**Examples:**

\[ x^2 = o(x) \quad \text{[} f(x) = x^2, \quad k(x) = x, \quad x \in \mathbb{R} \text{]} \]

\[ 2x + x^2 = O(x) \]

\[ \sqrt{x} \neq O(x) \]

\[ x^3 = o(x^2). \]

**Theorem** (The Chain Rule, stated in oh notation). If $f(a) = b$, and

\[ f(a + h) - f(a) = f'(a)h + o(h) \]

[where $h$ is the basic variable, $f'(a)$ is the matrix derivative]

and $g(b + k) - g(b) = g'(b)k + o(k)$,

then

\[ gf(a + h) - gf(a) = g'(b)f'(a)h + o(h). \]

Proof: Let $k(h) = f(a + h) - f(a)$, so $k(h) = f'(a)h + o(h) = O(h) + o(h) = O(h)$. Then
\[ g f(a + h) - g f(a) = g(b + k) - g(b) = g'(b)k + o(k) \]
\[ = g'(b)\left(f'(a)h + o(h)\right) + o(k) \]
\[ = g'(b)f'(a)h + g'(b)o(h) + o(O(h)) \]
\[ = g'(b)f'(a)h + o(h) + o(h) \]
\[ = g'(b)f'(a)h + o(h). \]

**Assignment.** Prove all the oh and Oh facts used in this argument.

*Remark:* One can also define \( o \) and \( O \) when \( x \) or \( k(x) \) has a limit other than 0. For instance, in discrete math and computer science, one considers \( f(n) \) as \( n \to \infty \). When the limit of interest is not obvious or fixed, one has to be explicit, and say things like “\( f(x) \) is little oh of \( x \) as \( x \to \infty \).”