More Problems for Week 8

Overview. These problems develop further the idea that properties can sometimes be transferred in a well-defined way from \( R^n \) (or some other well understood space) to other spaces. For instance, the idea of orientation can be transferred to any finite dimensional vector space, but the idea of right-handed orientation cannot. A transfer is well-defined if all the allowed ways to make the transfer give the same result. For instance, not all transfers of a frame from \( R^n \) to \( V \) agree on the sign of the frame, but they do all agree as to when two transferred frames have the same sign. Thus there is a well-defined concept in \( V \) of “same sign” frames, and thus all frames with the same sign are said to impart a common orientation. Put another way, same-sign is invariant under all allowed transfers. To my knowledge, proofs of invariance of transfer properties always have the following form. Given two transfer functions, \( S, T : R^n \rightarrow V \), you create the composition \( S^{-1}T \) in order to compare the results within the base space \( R^n \), where everything is already defined and you know what is going on.

None of the problems below require particularly hard computations. The hard part is figuring out what needs to be shown and recognizing it when you have shown it!

1. We continue with Problem 20.4 of Munkres, p177. David chose the linear isomorphism \( R^2 \rightarrow V \) defined by \( T(e_1) = a_1 \) and \( T(e_1) = a_2 \). With this choice, frame \( (a_1, a_2) \) in \( V \) is associated with frame \( (e_1, e_2) \) in \( R^2 \), which is a right-handed frame in \( R^2 \) since \( \det(e_1, e_2) = 1 > 0 \). Also with this choice, frame \( (a_3, a_4) \) in \( V \) is associated with frame \( ((1, 1), (1, -1)) \) in \( R^2 \), because \( T^{-1}(a_3) = (1, 1) \) and \( T^{-1}(a_4) = (1, -1) \); furthermore, \( ((1, 1), (1, -1)) \) is a left-handed frame, since \( \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 < 0 \). Thus we have that frames \( (a_1, a_2) \) and \( (a_3, a_4) \) have opposite orientation in \( V \), at least relative to \( T \). (By the way, I am writing vectors as rows for convenience in print; remember that they are columns in Munkres.)

Illustrate the theorem that orientation (but not right-handedness) is independent of \( T \) by redoing Problem 4 using the linear isomorphism \( \hat{T} \) defined by \( \hat{T}(e_1) = (-1, -1, 0) \) and \( \hat{T}(e_2) = (0, 0, 1/2) \). Verify that with \( \hat{T} \), the frames \( (a_1, a_2) \) and \( (a_3, a_4) \) continue to have opposite orientation, but that the first frame is no longer associated with a right-handed frame.

2. Chris had a different approach to Problem 20.4. Since \( V \subset R^3 \), he tried to create a frame that has a determinant by appending an additional vector to \( (a_1, a_2) \) and the same vector to \( (a_3, a_4) \) and comparing the signs of those two determinants. He too got that the signs were opposite.

However, I stopped him, because this approach does not correspond to the definition of orientation in general spaces \( V \) given on p171.

Still, his is a plausible way to define orientation when \( V \) is a subspace of some \( R^m \), so let’s pursue it. Maybe his idea has more merit than I gave it.

Definition. If \( B = (v_1, \ldots, v_n) \) is a frame (that is, ordered basis) of \( V \subset R^m \), let \( S \) be any ordered set of vectors \( u_{n+1}, \ldots, u_m \) such that \( C = (v_1, \ldots, v_n, u_{n+1}, \ldots, u_m) \) is a frame of \( R^m \). We say that \( B \) is right-handed relative to extension \( S \) if \( \det(C) \) is positive, and left-handed otherwise. We say that frames \( B \) and \( B’ \) of \( V \) have the same McKitterick orientation relative to \( S \) if they are both right-handed or both left-handed relative to \( S \).
a) Prove that McKitterick orientation is well-defined (i.e., independent of $S$) but that McKitterick right- and left-handedness is not.

b) When $V$ is an $n$-dimensional subspace of $R^m$, we now have two coherent definitions of orientation of frames, the Munkres definition and the McKitterick definition. Do they agree? That is, for every $V$, is the partition of frames into two sets according to Munkres the same partition obtained by McKitterick?

3. In this problem we consider vector spaces which don’t even have an inner product, so Munkres method of Problem 21.5 doesn’t apply to define volume of parallelepipeds. We show that we can nonetheless make some progress in defining volume; we can define *same volume*.

**Definition.** Given a parallelepiped $P$ in $V$, let $B = (v_1, \ldots, v_n)$ be a frame made up of the direction vectors of the sides of $P$. Similarly, for parallelepiped $P'$, let $B' = (v'_1, \ldots, v'_n)$ be a frame of its sides. Now let $T$ be an isomorphism of $R^n$ to $V$. We say that $P$ and $P'$ have the *same volume* relative to $T$ iff $|\det(T^{-1}(B))| = |\det(T^{-1}(B'))|$. We say that the *volume* of $P$ relative to $T$ is $|\det(T^{-1}(B))|$.

This definition makes sense because $T^{-1}(B)$ is a frame in $R^n$, not $V$, and thus *has* a determinant.

Prove: “same volume” is well defined (is independent of $T$) but “volume” is not.

*Note:* This problem is rather silly, since we can always *impose* an inner product on $V$ and then we could talk about volume as in Munkres 21.5. Also, the sets to which my volume concept apply are rather limited — just parallelepipeds. We’d like a volume concept that applies to all rectifiable sets, and we’d like it to make volume invariant, not just same volume invariant. And Munkres develops such a volume concept soon. But the point of this problem is to practice the idea of extending additional definitions from $R^n$ to other spaces and proving invariance; thus I thought up this simplified context to give a new example of this process.