1. On p137, at the start of the proof of Lemma 16.2, Munkres says it is not hard to find a countable collection of rectangles meeting conditions (1) and (2). Do this:

For each \( x \) in some \( A_i \in \mathcal{A} \), pick a cube \( C_y \subset A_i \) whose center \( y \) has all rational coordinates, whose width is rational, and such that \( x \in \text{Int} \ C_y \).

Explain how you know you can do this and why it provides a countable collection meeting (1)–(2).

2. Verify the claim on p138 of Munkres that for any collection of rectangles created to meet (1)–(3) of Lemma 16.2, for each point \( x \in \text{Bd} \ S \), every neighborhood of \( x \) will contain points from infinitely many of the sets \( C_1, C_2, \ldots \). Does this claim also hold for every collection of supports \( S_i \) meeting conditions (1)–(4) of a partition of unity in Thm 16.3?

3. Consider the function \( f(x) = \frac{\sin x}{x} \).

   a) Verify that the improper integral \( \int_0^\infty f(x) \) exists using the Calc I definition

   \[
   \int_a^\infty f(x) \, dx \text{ exists and equals } S \text{ if } \lim_{b \to \infty} \int_a^b f(x) \, dx = S.
   \]

   b) For this \( f \), does \( \int_0^\infty f(x) \, dx \) exist in Munkres sense? Prove your claim.

   The issue here is to give a complete argument. In seminar I indicated that this integral behaves like the alternating harmonic series. You’ve got to show how. (One of your readings this week shows a way to evaluate this integral exactly, but come up with your own analysis, which will probably be shorter and will probably only show the integral converges, not its value.)

4. Let

   \[
   f(x, y) = \frac{x - y}{(x + y)^3}. \tag{1}
   \]

   We wish to find \( \int_S f \) where \( S \) is the unit square \((0, 1)^2\). But \( f \) is improper on \( S \).

   a) Verify that

   \[
   f(x, y) = \frac{d}{dx} \frac{-x}{(x + y)^2} = \frac{d}{dy} \frac{y}{(x + y)^2}.
   \]

   b) Let \( U_n = (\frac{1}{n}, 1) \times (0, 1) \). By one of Munkres’ theorems, it is enough for us to find

   \[
   \lim_{n \to \infty} \int_{U_n} f(x, y), \text{ and since } f \text{ is bounded and continuous on each } U_n, \text{ by a bunch of other theorems (which ones?)}
   \]

   \[
   \int_{U_n} f(x, y) = \int_{1/n}^1 \int_0^1 f(x, y) \, dy \, dx.
   \]

   Evaluate the integral on the right (which is an ordinary iterated integral) and then let \( n \to \infty \). (We might call the result \( \int_0^1 \int_0^1 f(x, y) \, dy \, dx \).)

   c) Let \( V_n = (0, 1) \times (\frac{1}{n}, 1) \). Find \( \lim_{n \to \infty} \int_{V_n} f(x, y) \).

   (We might call the result \( \int_0^1 \int_0^1 f(x, y) \, dx \, dy \).)
d) Let \( W_n = (\frac{1}{n}, 1) \times (\frac{1}{n}, 1) \). Find \( \lim_{n \to \infty} \int_{W_n} f(x, y) \).
e) What’s wrong here? What morals do you draw from this problem?

5. Let \( f(x, y) = e^{-(x+y)} \). Suppose we wish to integrate this over the first quadrant \( R^2_+ \). One way you might have done this in Math 33 or 34 would be as

\[
\int_0^\infty \int_0^\infty f(x, y) \, dx \, dy. \tag{2}
\]

That is, you would do an iterated 1-dimensional improper integral. This amount to going to the limit of proper integrals twice, and is therefore, at least on the face of it, different from what Munkres does.
a) Evaluate (2) for this \( f \).
b) In this case, \( \int_{R^2_+} f \) exists in Munkres’ sense too and the values are equal. Prove this by as general a proof as you can. That is, you could prove it for this \( f \) by simply computing \( \int_{R^2_+} f \) by one of Munkres’ methods, but I am hoping you can devise a more general result that includes this \( f \) as a special case. Be careful: a completely general result is false. To wit,

\[
\text{Falsehood:} \quad \int_{R^2_+} f \text{ exists and equals } S \iff \int_0^\infty \int_0^\infty f(x, y) \, dx \, dy \text{ exists and equals } S.
\]

So whatever generality you claim, you better prove it. (If you don’t see any general result, just do a Munkres’ calculation.)

6. This problem is about functions from \( \mathbb{R} \) to \( \mathbb{R} \) and the (global) Lipschitz condition. (See Munkres, p160, Problem 3 for the definition; this is an easier problem than the problem there.)
a) Prove: if \( f' \) exists and is bounded, the \( f \) satisfies the Lipschitz condition.
b) Prove: if \( f' \) exists and is unbounded, then \( f \) does not satisfy the Lipschitz condition.

The same conclusions are true for functions from \( \mathbb{R}^m \) to \( \mathbb{R}^n \) but it takes more care to show them – the sort of care you have seen in Munkres many times. I limit the problem to real functions to focus on the main point. Also, a Lipschitz function doesn’t have to be differentiable. I can construct one that is nondifferentiable at every rational point. I don’t know if there is one that is nondifferentiable everywhere.