Tensors

Assume throughout these problems that the tensors are acting on \( R^n \) for some \( n \) and that the \( \phi \)'s are defined in terms of the standard basis \( e_1, \ldots \). Also assume (until further notice) that the coordinates of \( x \) are \( x_1, x_2, \ldots \), and likewise for \( y, w, \ldots \).

1. Write the following tensors in \( \phi \) notation. There should be no tensors products when you are done, i.e., \( \phi_1 \otimes \phi_3 \) should be replaced by something else.
   a) \( f(x, y) = x_3 y_4 - 3y_1 x_1 \)
   b) \( g(x, y, z, w) = f(x, y) f(z, w) \)
   c) \( f \otimes h \) where \( h(x, y) = x_2 y_1 + 2x_2 y_4 \)

2. Write the following tensors in the form of Problem 1; Munkres calls this “function form”.
   a) \( \phi_{31} \)
   b) \( \phi_4 \otimes \phi_2 \otimes \phi_1 \otimes \phi_4 \)
   c) \( (\phi_2 + 2\phi_1) \otimes (\phi_2 - \phi_4) \)
   d) \( (\phi_2 + 2\phi_1)(\phi_2 - \phi_4) \), where \( (fg)(x) \) means \( f(x)g(x) \); but then do you get a tensor?
   e) \( (\phi_2 + 2\phi_1)(x)(\phi_2 - \phi_4)(y) \) [Why do we need the \( x \) and \( y \) in this part?]

3. Is \( f(x, y, z) = 2x_2 y_3 \) a tensor?

4. Are the following tensors? If so, express them in standard \( \phi \) notation.
   a) \( f(x, y, z) = z_1 y_4 x_5 \)
   b) \( g(x, y) = \phi_{24}(y, 2x) \)

5. Express \( \psi_{14} \) in \( \phi \) notation. Same for \( \psi_{235} \).

Alternating Tensors

6. (For those who have not studied permutation groups, or want a review) Consider \( \sigma = (251643) \), by which I mean \( \sigma(1) = 2, \sigma(2) = 5 \) and so on (that is, the outputs are shown in order. Illustrate Lemma 27.2 on p228 and the definition just above it as follows:
   a) Determine if \( \sigma \) is odd or even by counting the number of inversions.
   b) Express \( \sigma \) as a composition of elementary permutations, and verify that (a) of the lemma holds (i.e., the parity of the number of elementary permutations you use is the same as the parity of the number of inversions)
   c) Express \( \sigma \) as a composition of transpositions, permutations that switch two elements, not necessarily adjacent. Part (d) implies that the parity of the number of transpositions will also be the same as the parity of the number of inversions.

7. Write a formula for
   a) \( \psi_{12}(x, y) \)
b) $\psi_{135}(x, z, w)$

8. Write the following products as a linear combination of terms. If the terms can be written using $\psi$’s, do so. If not, but they can be written in terms of $\phi$’s, do so. If it is easiest to write them in terms of products like $x_i y_j$, do that. In any event, there should be no $\otimes$ or $\wedge$ symbols in your final answer.

a) $(\psi_2 + 3\psi_3) \wedge (\psi_1 - 2\psi_3)$

b) $(\psi_2 + 3\psi_3) \otimes (\psi_1 - 2\psi_3)$

c) $(\psi_2 + 3\psi_3)(x)(\psi_1 - 2\psi_3)(y)$

d) Do the answers change if we replace each $\psi$ in the statement of a)–c) with the corresponding $\phi$?

9. In $\psi$ language, find a simple expression for

$$f = (\psi_{12} + \psi_{23})(2\psi_{14} - \psi_{34}).$$

Now assume the underlying vector space $V$ is $\mathbb{R}^4$. Give a determinant formula for $f$.

10. Consider the tensor $f = \phi_{431} - \phi_{341}$.

a) Add or subtract as many terms as necessary to $f$ to turn it into a $\psi$ (an elementary alternating tensor).

b) Directly from the definition compute $Af$ and then write the answer in $\psi$ form.

11. The definition of $f \wedge g$ about $3/4$ down p238 (an unnumbered equation!) makes perfectly good sense even for any tensors $f$ and $g$, alternating or not, and the product is alternating.

a) Compute $f = (\phi_{12} - 2\phi_{23}) \wedge \phi_{3}$

b) Why do you think Munkres restricted the definition of $\wedge$ to alternating forms?

12. Let us define $\psi_I$ for all ordered sets of indices from $\{1, 2, \ldots, n\}$ by $\psi_I = A\phi_I$, where $A$ is the averaging operator defined by Munkres at the bottom of p237.

a) Show that even with this extended definition, $\psi_I \wedge \psi_J = \psi_{I\&J}$, where $\&$ means concatenation. (You can henceforth use this extended $\psi_I$ notation wherever you want.) Hint: use display (*) on the bottom of p241.

b) Explain $\psi_I$ in terms of $\psi_I$ for sets $I$ where Munkres defined $\psi$.

13. Suppose all the $f$’s below are 1-forms, and that

$$g = f_{i_1} \wedge f_{i_2} \wedge \cdots \wedge f_{i_k} \quad \text{and} \quad g' = f'_{i_1} \wedge f'_{i_2} \wedge \cdots \wedge f'_{i_k}$$

are the same wedge product except that in the second product, the order of two (not necessarily adjacent) $f$’s is reversed. From the properties of the wedge product, state and prove a relationship between $g$ and $g'$. 

2
Forms

14. What is the difference between the $k$-tensor $\psi_I$ and the form $\tilde{\psi}_I$?

15. On p250, in going between the last two displays, in the double sum what justifies writing $b_I c_J \tilde{\psi}_I \wedge \tilde{\psi}_J$ instead of $b_I \tilde{\psi}_I \wedge c_J \tilde{\psi}_J$?

Ideally, there is some property displayed by Munkres that you can just point to (since he emphasizes properties).

16. Consider the form $\gamma$ on $\mathbb{R}^2$ defined by $\gamma = e^x - y \tilde{\psi}_{12}$. Evaluate this form at $(x; u, v) = (0; e_1, e_2)$

b) $(2, 1); (1, 1); (-2, 3))$. This is my shorthand for Munkres’ $\left((2, 1); (1, 1), (2, 1); (-2, 3)\right)$.

17. In Section 30 Munkres introduces (and explains) the notation

- $\tilde{\psi}_i = dx_i$
- $\tilde{\psi}_{12} = dx \wedge dy = dx_{12}$
- $\tilde{\psi}_I = dx_i \wedge \cdots \wedge dx_k = dx_I$

This gets a little dicey when we work with specific cases with low $n$, as in the middle line above, because it is natural to use $x, y, z$ for the coordinates of $x$ in this case instead of the $x_1, x_2, x_3$ we have used previously. But this conflicts with another convention we and Munkres used when introducing $k$-tensors for $k > 1$. Namely, $x, y, z$ were the different vectors making up the $k$-tuples in $V^k$ that the tensors acted on. In general in the last part of Munkres, this last use of $x, y, z$ will be abandoned, and you will have to tell what vector in the tangent space $dx_i$ acts on by its position in the wedge product.

a) (Warmup) With the new notation, $\gamma$ of Problem 16 becomes $\gamma = e^x - y dx \wedge dy$. Do that problem again, using the new notation. Then write a formula for $(dx \wedge dy)(u, v)$, where $u = (u_1, u_2, u_3)$ and likewise for $v$.

b) Let $\omega = x^2 dx$ and $\eta = y dx + x dy$ Let $x = (2, 1), u = (1, 1), v = (3, -1)$. At $(x; u)$ or $(x; u, v)$, whichever is appropriate, compute

a) $\omega$

b) $\eta$

c) $\theta = \omega \wedge \eta$

d) $\rho = \eta \wedge \omega$

Derivatives of Forms

18. What in general is a field, as defined by Munkres? What, if anything, is the difference between a tensor field and a form?
19. Munkres talks about wedge products of forms (e.g., $\omega \wedge \theta$ in the statement of Lemma 29.3 on p250 is the first use I can find) but as far as I can tell he never defines this product. What’s the definition?

20. Let $M$ be a 2-manifold (i.e., a nice surface) in $\mathbb{R}^3$. We consider fields and forms defined in some open set $U$ of $\mathbb{R}^3$ containing $M$. Consider the vector field

$$f(x) = (f_x(x), f_y(x), f_z(x)), \quad x \in U;$$

here $f_x$ does not indicate a partial derivative, but merely indicates the first coordinate of the vector $f(x)$.

Consider the form

$$\omega = f_x dy \wedge dz + f_y dz \wedge dx + f_z dx \wedge dy.$$ 

What does $\omega$ acting on $(x; u, v)$ represent?

*Hint:* Do you recognize the vector

$$((dy \wedge dz)(u, v), (dz \wedge dx)(u, v), (dx \wedge dy)(u, v))?$$

If not, compute it for $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$.

21. Let $f : \mathbb{R}^n \to R$ be differentiable. According to Munkres, what is the difference between the derivative function $Df$ and the differential form $df$?

22. Claim: $d\pi_i = \pi_i$. Reason: Both functions act on $n$-tuples and output the $i$th entry. Critique this claim.

23. a) In $xdx$ both $x$ and $dx$ pick out the first coordinate. So is this really $x^2$ in disguise? Explain.

   b) Similarly, in $xy^2dx \wedge dy$, both $x$ and $dx$ pick out the first coordinate, and $y^2$ and $dy$ pick out the second coordinate. Are they all picking coordinates from the same thing? Explain.

24. Since Munkres (and tradition) write $\tilde{\psi}_I$ as $dx_I$, it sounds plausible to declare $\tilde{\psi}_I$ to be $dx_I$. This is not the definition Munkres gave for $dx_I$. Is it equivalent?

25. Find $d(\tilde{\psi}_I)$. *Note:* We can’t just say that $d(\tilde{\psi}_I) = d(x_I) = 0$ by the double differential theorem, because Munkres is at pains to point out that $dx_I$ is not defined as a differential.

26. Show that $dx_I$ is a differential, even though Munkres didn’t define it that way.

27. Since $x_i$ is just the projection function (with an abusive name), it might make sense to define $x_I$ by

$$x_I = \prod_{i \in I} \pi_i = \prod_{i \in I} x_i$$

[Useful definition?]
28. Let $f : \mathbb{R}^n \to \mathbb{R}$ be $C^1$. Let $L_c = \{x | f(x) = c\}$ be an implicit surface and let $p \in L_c$. Assume that $\nabla f$ is never 0, so that $L_c$ is a $k = n - 1$ manifold. In multivariate calculus you (probably) decided that the tangent hyperplane to $L_c$ at $p$ is the hyperplane in $\mathbb{R}^n$ through $p$ normal to $Df(p)$. Translate this language into Munkres language about tangent spaces, and prove it in that context. Note: “Translate” may not just mean changing words; it may also involve changing the names of points.

29. Let $g : \mathbb{R}^k \to \mathbb{R}$ be $C^1$. In multivariate calculus you (probably) decided that at each point $(q, g(q))$ of the graph of $g$, there is a tangent hyperplane and it has the equation $T(x) = g(q) + Dg(q) \cdot (x - q)$. (That is, the tangent hyperplane is the graph of the 1st order Taylor approximation.)

In Munkres terms, the graph is an $n$-manifold in $\mathbb{R}^{n+1}$. That tangent hyperplane better turn out to be the tangent space to the manifold at $(q, g(q))$. Show that it is. That is, translate everything into Munkres manifold language and verify that we are talking about the right set.