When grappling with new concepts, I find it helpful to make up and solve problems about them. So here are the problems I made up today based on the presentations in seminar yesterday.

**Strong Derivatives.** Here is the definition of strong derivative for real functions. Kevin gave a version that works for vector functions also, but in grappling with new concepts, I find it helpful to work with the simplest case.

$f : \mathbb{R} \to \mathbb{R}$ is **strongly differentiable** at $a$, with derivative value $f^*(a)$, if

$$\lim_{x \neq x' \to a} \frac{f(x') - f(x)}{x' - x} = f^*(a).$$

The limit notation means that both $x, x'$ go to $a$ simultaneously, but are not equal. That is,

$$\forall \epsilon \exists \delta \ni |x' - a| < \delta \text{ and } |x - a| < \delta \text{ and } x \neq x' \implies \left| \frac{f(x') - f(x)}{x' - x} - f^*(a) \right| < \epsilon.$$  

1. Consider $f(x) = \frac{1}{2}x + x^2 \sin(1/x)$. (Kevin considered this function too, but for a slightly different purpose.)

   a) Show that $f'(0)$ exists and is positive.

   b) Show that, nonetheless, there is no neighborhood of 0 in which $f$ is an increasing function, and thus no neighborhood in which $f$ has an inverse.

   c) Why doesn’t b) contradict the Inverse Function Theorem?

   d) Suppose $g^*(0)$ is positive. Prove (without any other assumptions about $g$) that $g$ is a strictly increasing function in some neighborhood of 0. (This part gets at why strong derivatives can be said to be the right concept for inverse function theory.)

**Henstock Integrals.**

2. Prove: if $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then it is Henstock integrable with the same integral. **Note:** This follows immediately from the Theorem in Munkres problem 10.7, but say the one or two things that need to be said to make the Henstock integral definition fit what is stated in 10.7. How do you pick the Henstock $\delta$ function? How do the Henstock sums relate to the sums in 10.7?

3. Prove that the Henstock integral is well defined. That is, Henstock says that $\int_{[a,b]} f$ exists and equals $A$ if for every $\epsilon$ there exists a $\delta_\epsilon(x)$ function from $[a, b]$ to $(0, \infty)$ such that

$$\left| \sum f(t_i)(u_i - u_{i-1}) - A \right| < \epsilon \text{ whenever } t_i \in [u_{i-1}, u_i] \text{ and } u_i - u_{i-1} < d_\epsilon(t_i),$$  

where $a = u_0 < u_1 < \ldots < u_n = b$. But how do we know that there isn’t some other $B$ and some other set of delta functions $\delta'_\epsilon(x)$ such that (1) also holds using $B$ and $\delta'_\epsilon(x)$? (Unlike in
the Riemann case, we are not talking about the same set of approximating sums for the different 
\( \delta \)'s, and if \( f \) is unbounded, who knows what can happen with different approximations.) Show 
that this can't happen, i.e., that whenever the Henstock integral exists it is unique.

4. Consider the integral

\[
\int_0^1 \frac{1}{\sqrt{x}} \, dx.
\]  
(2)

Just so there is no quibbling about the integrand not being defined at the endpoint, define 
\( 1/\sqrt{0} = 0 \).

a) As a Riemann integral (2) is improper. Show this carefully. It’s obvious with the Darboux 
definition — why? — but show it’s also undefined with the definition of Riemann integrable 
in Munkres Problem 10.7.

b) Just for old times sake, compute the value of (2). (It’s tempting to just plug into the 
fundamental theorem of calculus, especially since this gives the right answer, but the 
method is wrong. Give a correct calculation. Then explain why the fundamental theorem 
gives the right answer.

c) Prove that (2) exists as a regular Henstock integral — no second limit to overcome im-
properness is needed. And show that the value is the same as in b). Hint: The first hard 
part is to find a \( \delta \) function and the second hard part is to show that (1) holds for your \( A \). 
Your \( \delta(x) \) function cannot be a constant independent of \( x \); why not? It took me a while 
to come up with a \( \delta(x) \) that works, and it’s messier than I would like. Maybe you can 
come up with a better one. But if you can’t, you can peek at mine — on the next page.
Hint for Problem 4c.

I define $\delta_\epsilon(x)$ as follows. First, note that $f(x) = 1/\sqrt{x}$ is a continuous function on $[a, 1]$ for any $a > 0$, and thus uniformly continuous on $[a, 1]$. Let $\delta'(\epsilon, a)$ be the $\delta$ such that, on the interval $[a, 1]$, $|f(x) - f(x')| < \epsilon$ whenever $|x - x'| < \delta$.

Next, noting that the improper Riemann integral (2) has value 2, let $\delta''(\epsilon)$ be the positive number such that

$$0 < a \leq \delta''(\epsilon) \implies 2 - \int_a^1 \frac{1}{\sqrt{x}} \, dx < \epsilon.$$  

We know such a $\delta''$ exists because the improper integral (a limit) exists.

Now define

$$\delta_\epsilon(x) = \begin{cases} 
\delta''(\epsilon/2) & \text{if } x = 0, \\
\min\{\frac{1}{2}x, \delta'(\frac{1}{2}\epsilon, \frac{1}{2}x)\} & \text{otherwise}.
\end{cases}$$

Then I claim that $|\sum f(t_i)(u_i - u_{i-1}) - 2| < \epsilon$, where the sum is any Henstock sum for gauge function $\delta_\epsilon$. 