Michael Stone’s Limit Argument

If I understand Michael’s argument, it was the following: Suppose for some function \( f \) you know that

\[
\lim_{x \to a} \lim_{y \to b} f(x, y) = L,
\]
that is, all required limits exist and the final limit is \( L \). Then we may pick any sequences \( x_k \to a \) and \( y_j \to b \) and reach the same limit. In particular, if \( b = a \) we may pick the sequences to be the same, say the sequence \( a_n \to a \). So

\[
\lim_{k \to \infty} \lim_{j \to \infty} f(a_k, a_j) = L. \tag{1}
\]

But since it’s the same sequence, we don’t need to use limits twice; instead we may write

\[
\lim_{k \to \infty} f(a_k, a_k) = L. \tag{2}
\]

Alas, this principle is incorrect. Here is a first reason to be suspicious. If this argument is correct, then also

\[
\lim_{j \to \infty} \lim_{k \to \infty} f(a_k, a_j) \tag{3}
\]

reduces to \( \lim_{k \to \infty} f(a_k, a_k) \) and thus we must have

\[
\lim_{j \to \infty} \lim_{k \to \infty} f(a_k, a_j) = \lim_{k \to \infty} \lim_{j \to \infty} f(a_k, a_j) \tag{4}
\]

But this says that the order of taking limits can be reversed, and in general that’s false.

Of course, the only proof that \( (1) \not\Rightarrow (2) \) is a counterexample. So I will start with a counterexample to \( (4) \), and sure enough, it provides a counterexample for Michael.

Let \( f(x, y) = x^{1/(1-y)} \). Let \( a_n = 1 - (1/n) \).

**Assignment.** For this \( f \) and this sequence \( \{a_n\} \), compute the limits in \( (1) \), then \( (3) \), then \( (2) \).

**Note:** My counterexample to \( (4) \) is based on my favorite counterexample (because it’s so simple) to the hope that limits commute. Namely:

\[
\lim_{x \to 1^-} \lim_{y \to \infty} x^y \neq \lim_{y \to \infty} \lim_{x \to 1^-} x^y.
\]

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