More Normed Metric Space Problems

Most of these problems are adapted from George Simmons, *Introduction to Topology and Modern Analysis*.

1. Let $B$ be a normed metric space. Let $B_\epsilon(x)$ be the open ball of radius $\epsilon$ around $x$.
   a) Suppose $|y - x| = \delta$. Prove that for every positive $\alpha < \delta + \epsilon$ there is a $z \in B_\epsilon(x)$ such that $|z - y| = \alpha$. Note: This should seem obvious once you draw a picture, but it is not true in all metric spaces. Consider the metric space in which all distinct points have distance 1. Or use the Euclidean metric on $\mathbb{R}^2$ but let the space consist of a random subset of $\mathbb{R}^2$.
   b) Prove that in every metric space, with $x, y$ as above, there exists no $z \in B_\epsilon(x)$ such that $|z - y| \geq \delta + \epsilon$. (Here we use $|a - b|$ as the metric, though in general the bar notation is reserved for norms.)

2. Let $N$ be a normed linear space (that contains more than the point zero). Prove that $N$ is a Banach space $\iff$ its unit sphere is a complete set (every Cauchy sequence within the set converges to a point in the set)

3. Let $M, N$ be vector spaces with norms $\| \cdot \|_M$ and $\| \cdot \|_N$. Define $M \oplus N$, the direct sum of $M, N$, as $\{(x, y) : x \in M, y \in N\}$, with addition and scalar multiplication defined coordinatewise. So far, this is just the standard linear algebra direct sum. Now define a norm on $M \oplus N$ by
   \[ \|(x, y)\| = \|x\|_M + \|y\|_N. \] (1)
   Prove: (1) really is a norm, and if $M, N$ are Banach spaces, so is $M \oplus N$.

4. Let $M, N$ be normed linear spaces, and let $L(M, N)$ be the set of all continuous linear transformations $T$ from $M$ to $N$, with addition and scalar multiplication of transformations defined as usual. As we know from Kolmogorov, a norm can be put on $L(M, N)$ by $\|T\| = \sup\{|T(x)| : |x| = 1\}$.
   a) Show that this norm really is a norm.
   b) Show: If $N$ is a Banach space, so is $L(M, N)$.

5. Let $B$ be a Banach space. Prove: $B$ is reflexive $\iff B^*$ is reflexive.