Difficulty with Sups, Part II

For the written homework assignment based on More Problems for Week 5, many of you chose to do problems 1 and 2 on integrability. Nothing wrong with that — these were good problems about handling limits, especially #1 — though I do hope you worked on the problems beyond the first two.

Anyway, a common error appeared on several of your papers, so I want to discuss it.

The problem arose when you proved from the definition that, if \( f \) and \( g \) are integrable, then so is \( f + g \). At one point you correctly showed that

\[
L(f+g, P) \geq L(f, P) + L(g, P). \tag{1}
\]

Then you immediately claimed that

\[
\int f + g \geq \int f + \int g. \tag{2}
\]

Since you gave no reason, presumably you “sup”ped both sides and then invoked the principle: If for all \( P \in \mathcal{P} \), \( c_P \geq a_P + b_P \), then

\[
\sup_{P \in \mathcal{P}} \{c_P\} \geq \sup_{P \in \mathcal{P}} \{a_P\} + \sup_{P \in \mathcal{P}} \{b_P\}. \tag{3}
\]

Unfortunately, this principle is false, even though (2) is true. The rest of this handout gets you to see why.

First, let’s break the false principle into two parts, one of which is true.

1. Prove: if \( C \) and \( D \) are sets of numbers, both indexed by set \( \mathcal{P} \), and

\[
c_P \geq d_P \quad \forall P \in \mathcal{P},
\]

then

\[
\sup C \geq \sup D.
\]

2. Prove that the following sup-addition property is false: If \( A \) and \( B \) are sets of numbers, both indexed by set \( \mathcal{P} \), then

\[
\sup_{P \in \mathcal{P}} \{a_P + b_P\} = \sup_{P \in \mathcal{P}} \{a_P\} + \sup_{P \in \mathcal{P}} \{b_P\}. \tag{falsehood 4}
\]

Note: That (4) is false can be very disconcerting, since the analogous statement

\[
\lim_{i \to \infty} \{a_i + b_i\} = \lim_{i \to \infty} \{a_i\} + \lim_{i \to \infty} \{b_i\}
\]

is true (at least if both limits on the right exist). But sups are not limits; they are bounds. Bounds work differently. In particular, even when all the sups in (4) exist (which is always the case if the sets are bounded), (4) is usually false.

3. Prove that false statement (4) can be made correct by making it an inequality, but that the inequality goes the wrong way for proving (2).
4. Prove: Despite the falseness of (4) for general sup addition, nonetheless

$$\sup_{P \in \mathcal{P}} \{L(f, P) + L(g, P)\} = \sup_{P \in \mathcal{P}} \{L(f, P)\} + \sup_{P \in \mathcal{P}} \{L(g, P)\}. \quad (5)$$

Of course, the truth of this must depend on the nature of partition sums, not just the nature of sups. So give a careful $\epsilon$ argument.

5. Complete the proof that (1) implies (2).

Remark: I said you made an error inferring (2) from (1). But (1) does imply (2), as we’ve now shown, so did you really make an error? Yes, you made an error of omission in that the argument can be completed but a reasonable reader would not regard it as complete as presented. You might say (even truthfully) that you meant all the steps we have now filled in but didn’t think you had to say them. But I feel you did have to say them, because in their absence one reasonably supposes you were assuming (3), since that is the only assumption which would make (2) follow immediately from (1). So you made at least an error of omission, and possibly a secret error of commission as well – an actual wrong step.

6. (Optional) If sups of sums are not done over indexed sets, but rather set sums are defined, then there is a sup addition law. For $A, B \subset \mathbb{R}$, define

$$A + B = \{a + b \mid a \in A, b \in B\}.$$ 

Prove

$$\sup(A + B) = \sup A + \sup B.$$