Problem:

Notation is the hardest part of generalizing Taylor’s Theorem to \( \mathbb{R}^n \). As you can see from this example third-order Taylor expansion of \( f(x) : \mathbb{R}^2 \to \mathbb{R}, f \in C^4 \) about \( a \), we observe multitudes of monomials:

\[
\begin{align*}
f(x) &= f(a) \\
&\quad + f'(1)(a)(x_1 - a_1) + f'(1)(a)(x_2 - a_2) \\
&\quad + (1/2!)(D_1^2 + D_2^2)f(a)(x_1 - a_1)^2 + D_1D_2f(a)(x_1 - a_1)(x_2 - a_2) \\
&\quad + (1/3!)(D_1^3 + D_2^3)f(a)(x_1 - a_1)^3 + D_1D_2D_3f(a)(x_1 - a_1)^2(x_2 - a_2) \\
&\quad + D_1D_2D_3f(a)(x_1 - a_1)(x_2 - a_2)^2 + D_2D_1D_2f(a)(x_1 - a_1)(x_2 - a_2)^2 \\
&\quad + D_2D_1D_2f(a)(x_1 - a_1)(x_2 - a_2)^2 + D_3D_1D_2f(a)(x_1 - a_1)(x_2 - a_2)^2 \\
&\quad + R_4(x; a)
\end{align*}
\]

Clearly, we find that we need to a good notation for expressing homogenous monomials of total degree \( k \).

Now, it happens that for first and second order Taylor expansions, matrix notation is adequate and leads to a nice classification of critical points. Unfortunately, to express higher-order expansions, we require higher-rank tensors, which no one seems to use in this context.

Instead, what people who want to express higher-order Taylor expansions of multivariate functions do is they use “multi-indices” to quickly develop all of the monomials they want to handle. Note: working in \( \mathbb{R}^n \), a “multi-index of order \( |k| \)” is just a fancy name for “partition of \( k \) into a sum of \( n \) non-negative integers”.

In order to use multi-indices, to express Taylor’s theorem, we need three more bits of notation. First, the setup. Let

\[
\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)
\]

where

\[
|\alpha| := \sum_{i=1}^{n} \alpha_i = k
\]

and where \( 0 \leq \alpha_i \leq k \) for each \( \alpha_i \). Let \( x \in \mathbb{R}^n \). Then

\[
x^\alpha := x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}.
\]

Conveniently, the same meaning extends to \( D^\alpha \). Finally, let

\[
\alpha! := \alpha_1\alpha_2!\cdots\alpha_n!.
\]

Then for \( f \in C^{n+1} \) on an open neighborhood \( U \) of \( a \in \mathbb{R}^n \), write

\[
P_k(x) = \sum_{|\alpha|=0}^{k} \frac{D^\alpha f(a)(x - a)^\alpha}{\alpha!}.
\]

Then one common formulation of Taylor’s Theorem states that the remainder term

\[
R_{k+1}(x; a) = f(x) - P_k(x) \in o_U(|x|^k).
\]

References:

Dieudonné, J. *Treatise on Analysis*, Volume IV. Section 19.5.

Bressoud, D. *A Radical Approach to Real Analysis*. Chapter 2.4.