1. Let $R_2^+$ be the First Quadrant: $\{(x, y) \mid 0 \leq x, y\}$. We want to verify that the double improper integral

$$\int_{R_2^+} e^{-xy} \sin x$$

(1)

can be iterated both ways, because one way ($y$ first) gives

$$\int_0^\infty \int_0^\infty e^{-xy} \sin x \, dy \, dx = \int_0^\infty \frac{\sin x}{x} \, dx, \quad (2)$$

which we want to evaluate, and the other way gives something we know:

$$\int_0^\infty \int_0^\infty e^{-xy} \sin x \, dx \, dy = \int_0^\infty \frac{1}{1 + y^2} \, dy = \lim_{b \to \infty} \tan^{-1} b - \tan^{-1} 0 = \pi/2. \quad (3)$$

Unfortunately, (1) is not absolutely integrable, so there is no theorem saying that the two iterated integrals are equal. If we are lucky, changing the order will give the same value in this particular instance, but we have to prove it.

So we look at

$$\int_{[0,N] \times [0,N]} e^{-xy} \sin x. \quad (4)$$

Since this is the integral of a continuous function over a bounded domain, it does equal the result of iterated integration either way:

$$\int_0^N \int_0^N e^{-xy} \sin x \, dy \, dx = \int_0^N \frac{\sin x}{x} - \frac{\sin x}{e^{N \pi} x} \, dx = \int_0^N \frac{1}{1 + y^2} - \frac{\cos N + y \sin N}{e^{N y} (1 + y^2)} \, dy. \quad (5)$$

So now we must take the limit as $N \to \infty$ of the right-hand sides of (5–6) and verify that we get the right-hand sides of (2) and (3).

Look carefully at the right side of (5):

$$\int_0^N \frac{\sin x}{x} \, dx - \int_0^N \frac{\sin x}{e^{N \pi} x} \, dx \quad (7)$$

and the right side of (6):

$$\int_0^N \frac{1}{1 + y^2} - \int_0^N \frac{\cos N + y \sin N}{e^{N y} (1 + y^2)} \, dy. \quad (8)$$

Clearly

$$\lim_{N \to \infty} \int_0^N \frac{\sin x}{x} \, dx = \int_0^\infty \frac{\sin x}{x} \, dx$$

by definition, once we know that the limit exists (see the proof it exists in Edwards). Also, we already know that

$$\lim_{N \to \infty} \int_0^N \frac{1}{1 + y^2} = \pi/2.$$
a) So what you have to prove is

\[
\lim_{N \to \infty} \int_{0}^{N} \frac{\sin x}{e^{N x} x} \, dx = 0
\]  

(9)

and

\[
\lim_{N \to \infty} \int_{0}^{N} \frac{\cos N + y \sin N}{e^{N y} (1 + y^2)} \, dy = 0.
\]  

(10)

Do it. *Hint:* In (9) use the fact that \(|(\sin x)/x| \leq 1\) for all \(x \geq 0\). In (10), begin by proving

\[
\left| \frac{\cos N + y \sin N}{1 + y^2} \right| \leq 2 \quad \text{for all } y \geq 0,
\]

or some other finite bound instead of 2.

b) In asserting (1–2), we assumed that Edwards was right and

\[
\int_{0}^{\infty} e^{-xy} \sin x \, dy = \frac{\sin x}{x},
\]  

(11)

and

\[
\int_{0}^{\infty} e^{-xy} \sin x \, dx = \frac{1}{1 + y^2}.
\]  

(12)

Verify that these 1st-year improper integrals are correct. Or are they? The inner integrals in (5–6) can help (they were gotten with *Mathematica*).

c) In b) you should have found that there is a problem with (11-12); there is exactly one value of \(x\) for which (11) is false, and exactly one value of \(y\) for which (12) is false. Which? Why?

Are there also values for which (5–6) are false? The bottom line is still correct: \(\int_{0}^{\infty} (\sin x/x) \, dx = \pi/2\). How come?