Theorem. Suppose \( f : X \to Y \), where \( X = A \cup B \) and \( A, B \) are closed in \( X \). Suppose further that both \( f|_A \) and \( f|_B \) are continuous. Then \( f \) is continuous.

I like Yusra’s inverse closed set proof better, because it is so short and because it generalizes easily to the case where \( X \) is a finite union of closed sets. However, there is value in showing White’s proof in more detail.

In what follows, \( d(x', x) \) and \( B(x, \delta) \) have the same meaning as in my solutions to Problem 3.5.

Proof: Pick \( x \in X \). Either \( x \in A \) or \( x \in B \). For the first case, we will show that \( f \) (not just \( f|_A \)) is continuous at \( x \). The same argument works for \( B \) (not repeated), showing that \( f \) is continuous everywhere.

So pick any \( \epsilon > 0 \) and consider two subcases.

Case 1a: \( x \in \text{Int}(A) \) (meaning interior in the topology of the whole space \( X \)). Since \( f|_A \) is continuous, \( \exists \delta \) such that

\[
(x' \in A \text{ and } d_X(x', x) < \delta) \implies d_Y(f(x'), f(x)) < \epsilon.
\]

Now pick \( \delta' \) so that \( B(x, \delta') \subset A \). We can do this since \( x \in \text{Int}(A) \). Set \( \delta'' = \min\{\delta, \delta'\} \). Now

\[
(x' \in X \text{ and } d_X(x', x) < \delta'') \iff (x' \in A \text{ and } d_X(x', x) < \delta'').
\]

\[
\iff (x' \in A \text{ and } d_X(x', x) < \delta) \implies d_Y(f(x'), f(x)) < \epsilon.
\]

Thus \( f \) is continuous at \( x \).

Case 1b: \( x \notin \text{Int}(A) \). Since the complement of \( A \) in space \( X \) is contained in \( B \), there must be points of \( B \) in every neighborhood of \( x \). Since \( B \) is closed, this means \( x \in B \).

Since \( x \in A \), by the continuity of \( f|_A \) there exists \( \delta \) such that

\[
(x' \in A \text{ and } d_X(x', x) < \delta) \implies d_Y(f(x'), f(x)) < \epsilon.
\]

Since \( x \in B \), by the continuity of \( f|_B \) there exists \( \delta' \) such that

\[
(x' \in B \text{ and } d_X(x', x) < \delta') \implies d_Y(f(x'), f(x)) < \epsilon.
\]

Set \( \delta'' = \min\{\delta, \delta'\} \). Then \( (x' \in X = A \cup B \text{ and } d_X(x', x) < \delta'') \) implies

either \( (x' \in A \text{ and } d_X(x', x) < \delta) \) or \( (x' \in B \text{ and } d_X(x', x) < \delta') \).

Either condition implies \( d_Y(f(x'), f(x)) < \epsilon \), so \( f \) is continuous at \( x \). ■