Mathematical Modeling Techniques
Difference Equations

1st-Order Linear Difference Equation

A first order linear difference equation has the form,

$$x_{n+1} = ax_n + b.$$

It depends only on the previous generation. Solutions are usually found iteratively. Find the first few generations. Once a pattern is established, write the solution.

Example 1: $x_{n+1} = ax_n$
Let x_0 be your initial condition. It can stand for any constant that you choose. Then,

$$\begin{aligned} x_1 &= ax_0 \\ x_2 &= ax_1 = a^2x_0 \\ x_3 &= ax_2 = a^3x_0 \\ &\vdots \\ x_n &= a^nx_0 \end{aligned}$$

2nd-Order Linear Difference Equation

A second order linear difference equation has the form,

$$x_{n+1} = ax_n + bx_{n-1}. \tag{1}$$

It depends on the previous two generations. Note, it could be written in the form

$$x_{n+2} = ax_{n+1} + bx_n. \tag{2}$$

(1) and (2) are the same equation!! Solutions can be found in this form (i) or by converting to a system of first order linear difference equations (ii).

- i Solutions have the form

  $$x_n = A_1\lambda_1^n + A_2\lambda_2^n$$

  where $\lambda_{1,2}$ are solutions of

  $$\lambda^2 - a\lambda - b = 0.$$

- ii To solve by converting to a system of first order linear difference equations, let $x_{n-1} = y_n$. Therefore, $y_{n+1} = x_n$ and (1) becomes

  $$\begin{aligned} x_{n+1} &= ax_n + by_n \qquad (3) \\ y_{n+1} &= x_n \qquad (4) \end{aligned}$$

  See the section on Systems of 1st Order Linear Equations for details on how to solve (3)-(4). Note: the solution will be the same as (i).

Systems of 1st Order Linear Equations

Regardless of how many variables (populations) are in the model (equations), each variable in the generation only depends on the variables in the previous generation. This is why they are called first order. Equations have the form,

$$
\begin{aligned}
x_{n+1} &= a_{11}x_n + a_{12}y_n && (5)\\
y_{n+1} &= a_{21}x_n + a_{22}y_n && (6)
\end{aligned}
$$

Since (5)-(6) can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix}_{n+1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_n$$

Solutions have the form

$$
\begin{aligned}
x_n &= A_1\lambda_{1^2} + A_2\lambda_2^2\\
y_n &= B_1\lambda_1^2 + B_2\lambda_1^2
\end{aligned}
$$

where $\lambda_{1,2}$ are solutions of the eigenvalue problem

$$det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} = 0,$$

and $A_{1,2}$, $B_{1,2}$ are found from the initial conditions provided. If no initial conditions are provided, you can leave as $A_{1,2}$ and $B_{1,2}$.

Linear Algebra Note (not required for class - just making connections): We could write (7)-(7) as $\vec{z}_n = \vec{v}_1\lambda_1 + \vec{v}_2\lambda_2$ *where*

$$\vec{z}_n = \begin{pmatrix} x \\ y \end{pmatrix}_n$$

$\lambda_{1,2}$ *are the eigenvalues and* $\vec{v}_{1,2}$ *are the eigenvectors. Note: the eigenvectors are scaled for the initial conditions.*

1st Order Non-Linear Difference Equations

First order non-linear difference equations have the form

$$x_{n+1} = f(x_n).$$

The population in a generation only depends on the previous generation, even though the terms can be quite complicated. We do not solve non-linear difference equations. Therefore, the dynamics depend upon 'end-behavior' not 'solutions'. To explore the end-behavior, find steady states and stability.

Steady states are when the population is constant over generations,

$$x_{n+1} = x_n = \bar{\mathrm{x}}.$$

All steady states must be non-negative to be biologically accurate!! Be sure to find parameter conditions for the steady states to be non-negative! We can have multiple steady states for the model.

To determine whether solutions that start near the steady states will approach it, we determine stability. Stability for each steady state is found by testing

$$|f'(\bar{\mathrm{x}})| < 1.$$

Systems of Non-Linear Difference Equations

Once again, the population in a generation only depends on the populations in the previous generation. Systems of non-linear difference equations have the form,

$$\begin{aligned} x_{n+1} &= f(x_n, y_n), \\ y_{n+1} &= g(x_n, y_n). \end{aligned}$$

Since we can't solve these types of models, we must find steady states and try to determine the stability of each steady state.

Steady states are defined by a population staying constant. Therefore, they are found by setting

$$\begin{aligned} x_{n+1} = x_n = \bar{\mathrm{x}} \\ y_{n+1} = y_n = \bar{\mathrm{x}} \end{aligned}$$

and solving the system of equations. We can have multiple steady states, but they must be non-negative. Finding the parameter regimes for the steady state to be non-negative can help with stability conditions.

Stability is determined by taking the partial derivatives of $f(x_n, y_n)$ and $g(x_n, y_n)$.

$$\begin{aligned} a_{11} &= \frac{\partial f}{\partial x_n}|_{(\bar{\mathrm{x}},\bar{\mathrm{y}})} \\ a_{12} &= \frac{\partial f}{\partial x_n}|_{(\bar{\mathrm{x}},\bar{\mathrm{y}})} \\ a_{21} &= \frac{\partial g}{\partial x_n}|_{(\bar{\mathrm{x}},\bar{\mathrm{y}})} \\ a_{22} &= \frac{\partial g}{\partial x_n}|_{(\bar{\mathrm{x}},\bar{\mathrm{y}})} \end{aligned}$$

For stability, $|\lambda_{1,2}| < 1$ where $\lambda_{1,2}$ are given by

$$\begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix}$$

Since we are only concerned with stability and not solutions, we don't need the actual λ values, we just need both of their magnitudes to be less than 1. For this to hold,

$$|\beta| < 1 + \gamma < 2 \tag{7}$$

where

$$\begin{aligned} \beta &= a_{11} + a_{22} \\ \gamma &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$