Mathematical Modeling
Bifurcation Diagrams / Bifurcation Points

Bifurcation diagrams are helpful to understand how the long term behavior of a model changes as parameters change. Points on the diagram that represent changes in the behavior are called bifurcation points. There are several different types of bifurcation points.

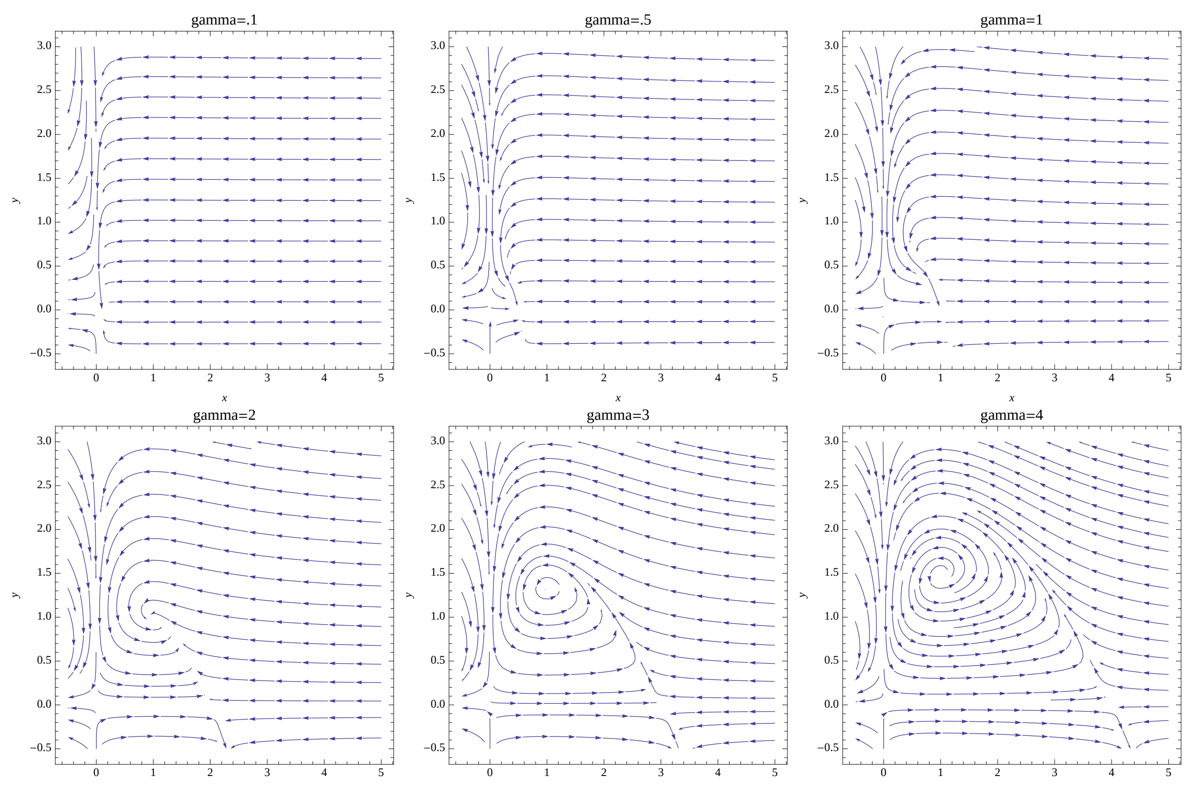
1. *Hopf Bifurcation Point*

   (a) Find the nullclines and steady states of the following model.

$$\begin{aligned}\frac{dx}{dt} &= x\left(1-\frac{x}{\gamma}\right)-\frac{xy}{1+x}\\ \frac{dy}{dt} &= \beta\left(\frac{x}{1+x}-\alpha\right)y\end{aligned}$$

   (b) Determine the stability of the steady states.

   (c) Construct the nullclines on the following phase planes.



   (d) Construct a bifurcation diagram for x with respect to γ, varying γ from .1 to 5.

   (e) Construct a bifurcation diagram for y with respect to γ, varying γ from .1 to 5.

   (f) At $\gamma \approx 3$, there is a Hopf bifurcation point. Describe what this is.

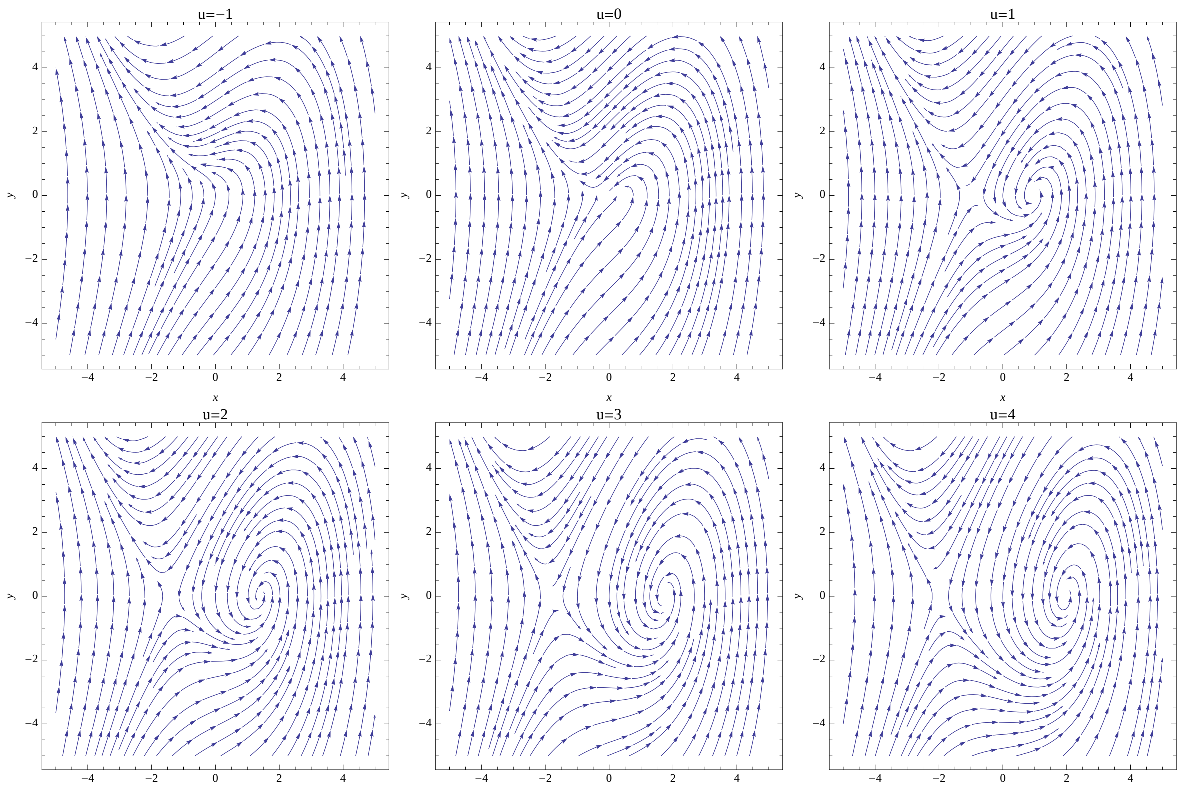
2. *Saddle-node Bifurcation Point*

(a) Find the nullclines and steady states of the following model.

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x^2 - y - \mu$$

(b) Determine the stability of the steady states.

(c) Construct the nullclines on the following phase planes.



(d) Construct a bifurcation diagram for x with respect to μ, varying μ from -1 to 4.

(e) Construct a bifurcation diagram for y with respect to μ, varying μ from -1 to 4.

(f) At $\mu = 0$, there is a saddle-node bifurcation point. Describe what this is.

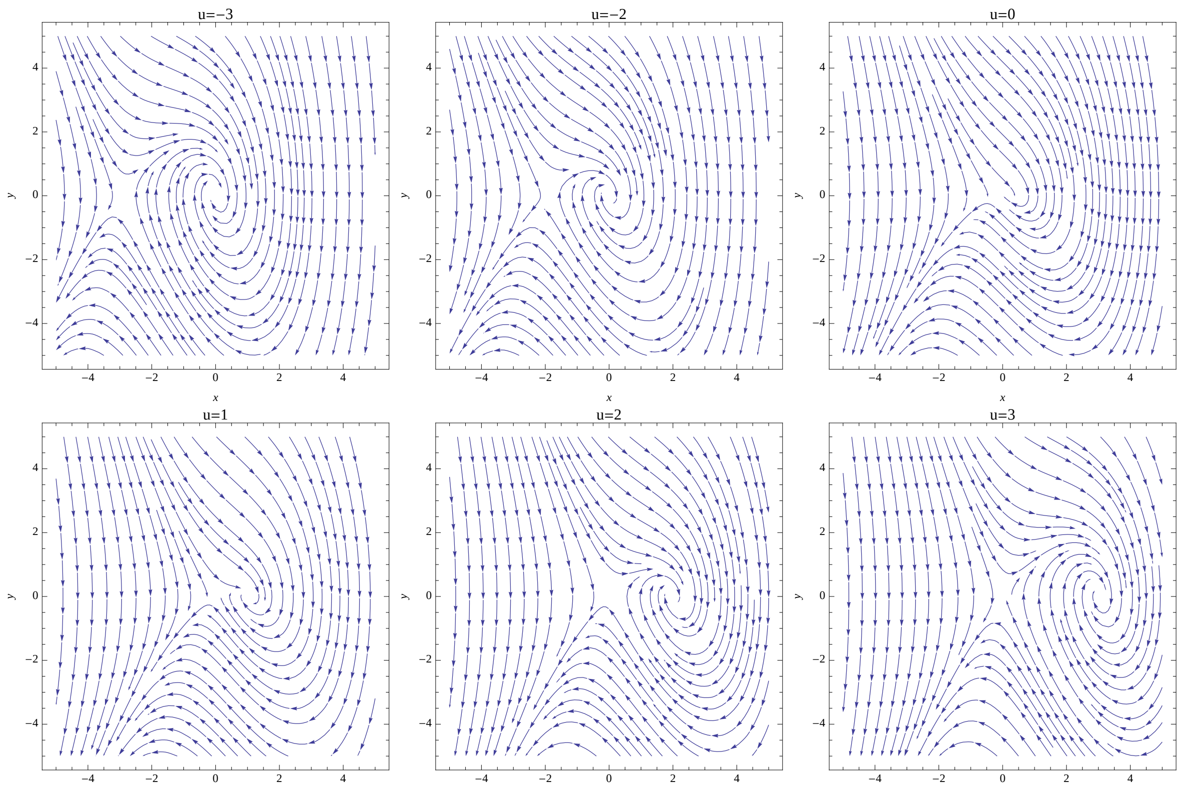
3. *Transcritical Bifurcation Point*

(a) Find the nullclines and steady states of the following model.

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = \mu x - x^2 - y$$

(b) Determine the stability of the steady states.

(c) Construct the nullclines on the following phase planes.



(d) Construct a bifurcation diagram for x with respect to μ, varying μ from -3 to 3.

(e) Construct a bifurcation diagram for y with respect to μ, varying μ from -3 to 3.

(f) At $\mu = 0$, there is a transcritical bifurcation point. Describe what this is.

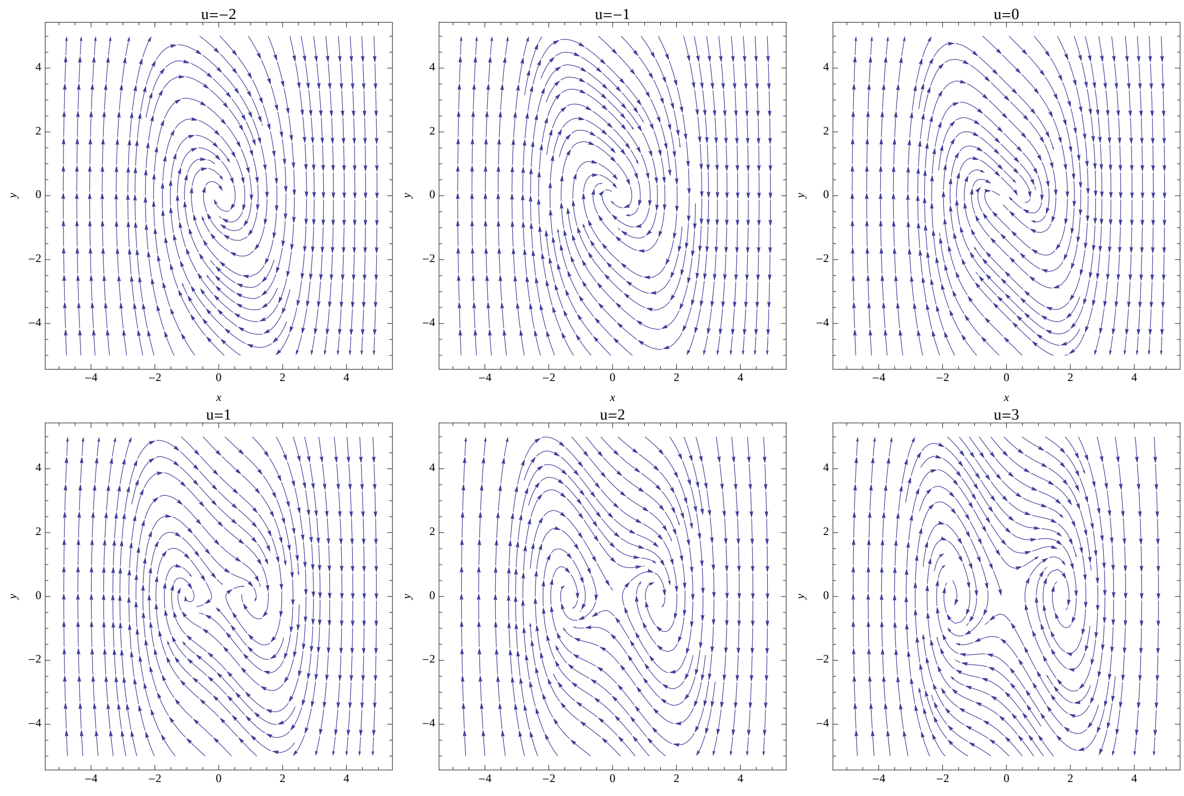
4. *Pitchfork (or flip) Bifurcation Point*

(a) Find the nullclines and steady states of the following model.

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \mu x - x^3 - y\end{aligned}$$

(b) Determine the stability of the steady states.

(c) Construct the nullclines on the following phase planes.



(d) Construct a bifurcation diagram for x with respect to μ, varying μ from -3 to 3.

(e) Construct a bifurcation diagram for y with respect to μ, varying μ from -3 to 3.

(f) At $\mu = 0$, there is a pitchfork bifurcation point. Describe what this is.