Mathematical Modeling Homework
Week 8

Complete problems 1 and 7 and 2 of the following: 2,3,4,5,6.

1. Read "The Case of the Missing Meat Eaters," *Natural History*, June 1993, 22-24. The predator situation in Australia as described in Flannery's article might be modeled by the following differential equation for $k(t)$ = number of kangeroos at time t (the prey) and $p(t)$ = number of predators at time t:

$$\frac{dk}{dt} = \alpha k - \beta k^2 - \gamma kp,$$
$$\frac{dp}{dt} = -\sigma p + \lambda kp,$$

where $\alpha,\ \beta,\ \gamma,\ \sigma,$ and λ are positive constants.

   (a) Does the model use unbounded or bounded growth for the prey? If the growth is unbounded, what is the growth rate? If the growth is bounded, what is the maximum growth rate and carrying capacity?

   (b) What parameter will change if the lifespan of the predator is changed? How will the value change for a short lived predator vs. a long-lived predator.

   (c) What parameter represents the efficiency that an animal utilizes food to make babies? How will the value change if the animal uses food efficiently vs. inefficiently.

   (d) Mathematically analyze the model.

   (e) Analyze the model using nullclines and pplane (include necessary figures for both cases).

   (f) Biological explain results from (d)-(e).

   (g) Are there periodic orbits in the first quadrant of the model? Prove your result.

2. On page 223, a number of modifications of the Lotka-Volterra equations are described. Explain these modifications, paying particular attention to their predictions for low and high values of the prey population x.

3. On page 223, select a modification that we did not discuss in class. Assume it is the only change made to the original Volterra equations that we discussed in class and determine the effect on steady states and their stability properties.

4. Suppose that prey have a refuge from predators into which they can retreat. Assume the refuge can hold a fixed number of prey. How would you model this situation, and what predictions can you make?

5. Species may derive mutual benefit from their association; this type of interaction is known as mutualism. The following set of equations can describe a possible pair of mutualists,

$$\begin{aligned}\frac{dN_1}{dt} &= rN_1\frac{1-N_1}{k_1+\alpha N_2},\\ \frac{dN_2}{dt} &= rN_2\frac{1-N_2}{k_2+\beta N_1},\end{aligned}$$

where N_i is the population of the ith species, and $\alpha\beta < 1$.

(a) Explain why the equations describe a mutualistic interaction.

(b) Determine the qualitative behavior of this model by phase-plane and linearization methods.

(c) Why is it necessary to assume that $\alpha\beta < 1$?

6. Suppose a one-time fishing expedition reduced the prey population by 10% of its current level. What does the Lotka-Volterra model predict about the subsequent behavior of the system?

7. Read the Routh-Hurwitz Criteria in the box on Page 233. Use the following model from HW 7,

$$\begin{aligned}\frac{dx_1}{dt} &= D-(u-k_{12})x_1+k_{21}x_2,\\ \frac{dx_2}{dt} &= k_{12}x_1-(k_{21}+k_{23}+s)x_2+k_{32}x_3,\\ \frac{dx_3}{dt} &= k_{23}x_2-k_{32}x_3.\end{aligned}$$

Since the system is linear, there is only one steady state ($\bar{x}_1$,$\bar{x}_2$,$\bar{x}_3$). Determine the stability of the steady state using the following steps.

(a) Find the Jacobian at the steady state.

(b) Find the characteristic equation of the form $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$.

(c) Since the characteristic equation is of order 3, find H_1, H_2, and H_3. Note: $a_4 = a_5 = 0$.

(d) Find the determinant of each H and determine the condition for stability. Note: for stability, the determinant of all of the matrices (H_1, H_2, and H_3) have to be greater than zero.

8. The neutral stability of Volterra's predator-prey model is a consequence of the particular assumptions for the interaction between prey and predator. There have been many extensions toward increased realism, such as including maximum carrying capacities for prey and predators and saturating functional responses of predator to prey. All such extensions tend to remove the unrealistic neutral stability. Instead, the abundance of predators and prey will usually converge to either a stable equilibrium or a stable limit cycle. In contrast to neutral oscillations, a stable limit cycle is a robust phenomenon. After small perturbations, the trajectory returns to the limit cycle. In 1936, the Russian mathematician A.N. Kolmogorov wrote a general predator-prey equation in the following form,

$$\begin{aligned} \frac{dx}{dt} &= xF(x,y), \\ \frac{dy}{dt} &= yG(x,y). \end{aligned}$$

Here, F and G are continuous functions with continuous first derivatives. Kolmogorov's theorem states that the model has either a stable limit cycle or a stable equilibrium if a set of conditions hold. Biologically interpret each of the conditions below.

(a) $\frac{\partial F}{\partial y} < 0$

(b) $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} < 0$

(c) $\frac{\partial G}{\partial y} < 0$

(d) $x\frac{\partial G}{\partial x} + y\frac{\partial G}{\partial y} > 0$

(e) $F(0,0) > 0$

(f) There must exist a $A > 0$, $B > 0$, $C > 0$ s.t. $F(0,A) = 0$, $F(B,0) = 0$, and $G(C,0) = 0$.