

## Proof of the Central Limit Theorem

Suppose  $X_1, \dots, X_n$  are i.i.d. random variables with mean 0, variance  $\sigma_x^2$  and Moment Generating Function (MGF)  $M_x(t)$ . Note that this assumes an MGF exists, which is not true of all random variables.

Let  $S_n = \sum_{i=1}^n X_i$  and  $Z_n = S_n/\sqrt{n\sigma_x^2}$ . Then

$$M_{S_n}(t) = (M_x(t))^n \text{ and } M_{Z_n}(t) = \left( M_x\left(\frac{t}{\sigma_x\sqrt{n}}\right) \right)^n.$$

Using Taylor's theorem, we can write  $M_x(s)$  as

$$M_x(s) = M_x(0) + sM'_x(0) + \frac{1}{2}s^2M''_x(0) + e_s,$$

where  $e_s/s^2 \rightarrow 0$  as  $s \rightarrow 0$ .

$M_x(0) = 1$ , by definition, and with  $E(X_i) = 0$  and  $Var(X_i) = \sigma_x^2$ , we know  $M'_x(0) = 0$  and  $M''_x(0) = \sigma_x^2$ . So

$$M_x(s) = 1 + \frac{\sigma_x^2}{2}s^2 + e_s.$$

Letting  $s = t/(\sigma_x\sqrt{n})$ , we have  $s \rightarrow 0$  as  $n \rightarrow \infty$ , and

$$M_{Z_n}(t) = \left( 1 + \frac{\sigma_x^2}{2} \left( \frac{t}{\sigma_x\sqrt{n}} \right)^2 + e_n \right)^n = \left( 1 + \frac{t^2}{2n} + e_n \right)^n,$$

where  $n\sigma_x^2 e_n/t^2 \rightarrow 0$  as  $n \rightarrow \infty$ .

If  $a_n \rightarrow a$  as  $n \rightarrow \infty$ , it can be shown that

$$\lim_{n \rightarrow \infty} \left( 1 + \frac{a_n}{n} \right)^n = e^a.$$

It follows that

$$\lim_{n \rightarrow \infty} M_{Z_n}(t) = \lim_{n \rightarrow \infty} \left( 1 + \frac{t^2/2 + ne_n}{n} \right)^n = e^{t^2/2},$$

which is the MGF of a standard Normal. In other words,  $Z_n \xrightarrow{L} N(0, 1)$  as  $n \rightarrow \infty$  ( $Z_n$  converges in distribution to  $N(0, 1)$ ).

The practical application of this theorem is that, for  $Y_1, \dots, Y_n$  i.i.d. with mean  $\mu_y$  and variance  $\sigma_y^2$ , if  $n$  is "large", then

$$\sum_{i=1}^n \left( \frac{Y_i - \mu_y}{\sigma_y\sqrt{n}} \right) \sim N(0, 1), \text{ or } \bar{Y} \sim N(\mu_y, \sigma_y^2/n).$$

How large is "large" depends on the distribution of the  $Y_i$ 's. If Normal, then  $n = 1$  is large enough. As the distribution becomes less Normal, larger values of  $n$  are needed.