Proof of the Central Limit Theorem

Suppose $X_1, \ldots, X_n$ are i.i.d. random variables with mean 0, variance σ_x^2 and Moment Generating Function (MGF) $M_x(t)$. Note that this assumes an MGF exists, which is not true of all random variables.

Let $S_n = \sum_{i=1}^n X_i$ and $Z_n = S_n/\sqrt{n\sigma_x^2}$. Then

$$M_{S_n}(t) \;=\; (M_x(t))^n \;\text{ and }\; M_{Z_n}(t) \;=\; \left(M_x\left(\frac{t}{\sigma_x\sqrt{n}}\right)\right)^n.$$

Using Taylor's theorem, we can write $M_x(s)$ as

$$M_x(s) \;=\; M_x(0) + sM_x'(0) + \frac{1}{2}s^2 M_x''(0) + e_s,$$

where $e_s/s^2 \to 0$ as $s \to 0$.

$M_x(0) = 1$, by definition, and with $E(X_i) = 0$ and $Var(X_i) = \sigma_x^2$, we know $M_x'(0) = 0$ and $M_x''(0) = \sigma_x^2$. So

$$M_x(s) = 1 + \frac{\sigma_x^2}{2}s^2 + e_s.$$

Letting $s = t/(\sigma_x\sqrt{n})$, we have $s \to 0$ as $n \to \infty$, and

$$M_{Z_n}(t) \;=\; \left(1 + \frac{\sigma_x^2}{2}\left(\frac{t}{\sigma_x\sqrt{n}}\right)^2 + e_n\right)^n \;=\; \left(1 + \frac{t^2}{2n} + e_n\right)^n,$$

where $n\sigma_x^2 e_n/t^2 \to 0$ as $n \to \infty$.

If $a_n \to a$ as $n \to \infty$, it can be shown that

$$\lim_{n\to\infty}\left(1 + \frac{a_n}{n}\right)^n \;=\; e^a.$$

It follows that

$$\lim_{n\to\infty} M_{Z_n}(t) \;=\; \lim_{n\to\infty}\left(1 + \frac{t^2/2 + ne_n}{n}\right)^n \;=\; e^{t^2/2},$$

which is the MGF of a standard Normal. If the MGF exists, then it uniquely defines the distribution. Convergence in MGF implies that Z_n converges in distribution to $N(0,1)$.

The practical application of this theorem is that, for large n, if $Y_1, \ldots, Y_n$ are independent with mean μ_y and variance σ_y^2, then

$$\sum_{i=1}^n \left(\frac{Y_i - \mu_y}{\sigma_y\sqrt{n}}\right) \dot{\sim} N(0,1), \quad \text{or} \quad \bar{Y} \dot{\sim} N(\mu_y, \sigma_y^2/n).$$

How large is "large" depends on the distribution of the Y_i's. If Normal, then $n = 1$ is large enough. As the distribution becomes less Normal, larger values of n are needed.