Stat 111 Spring 2011 Week 1

Discrete Probability Problems

1. 10 raters are each asked to read 10 essays and choose the best one. To save time, the raters each choose an essay at random, independently of one another (so they might, for example, all choose the same essay).

   a) What is the probability that all 10 essays are chosen by a rater?

   b) What is the expected proportion of essays chosen by at least one rater?

   c) Answer b if n raters each choose one of m essays.

   d) If n and m are both very large, show that the expected proportion chosen is approximately $1 - e^{-n/m}$.

2. A radioactive source emits alpha particles according to a Poisson process with rate λ emissions per minute. Let X represent the number of particles emitted in a particular minute. To derive the Poisson probability function, think of X as approximately $B(n, p)$, where p is the probability of at least one emission in a time interval of length $1/n$ minutes (e.g., if $n = 60$, p is the probability for one second). For any finite n this distribution is only approximate because the count for an individual time interval could be greater than 1 (i.e., X is not the sum of n independent Bernoulli random variables). Show that in the limit as $n \to \infty$, with $np \to \lambda$, the $B(n, p)$ probability function converges to the Poisson(λ) probability function.

3. Suppose n graduates all throw their caps in the air and then retrieve a cap at random. Let X be the number of people who recover their own cap. Work out the distribution of X for $n = 2$ and $n = 3$. Derive a general formula for $P(X = 0)$ and show that this approaches $1/e$ (≈ 0.37) as $n \to \infty$. Hint: For $i = 1, \dots, n$, let A_i represent the event that person i gets her or his own cap. Then $P(X = 0) = 1 - P(A_1 \mathrm{U} A_2 \mathrm{U} \dots \mathrm{U} A_n)$, which can be expanded using the general rule for unions. Also, for large n the distribution of X is approximately Poisson(1) (why?).

4. In a given day, N people buy a scratch-off lottery ticket at a particular store. Assume the outcomes for the tickets are independent and that each has probability p of being a winner. Let X_1 be the number of winners and $X_2 = N - X_1$ be the number of losers in a given day. If $N \sim$ Poisson(λ), find the marginal distributions for X_1 and X_2.

5. Suppose the numbers of goals scored by team 1 and team 2 are independent Poisson random variables with rates λ_1 and λ_2. Let $N = X_1 + X_2$ be the total number of goals scored by team 1 and team 2. If we learn that $N = n$ goals were scored, what is the conditional distribution of X_1, the number of goals scored by team 1?

6* In the context of the previous problem, what is the probability that the *next* goal is scored by team 1? Show that the probability distribution of the number of goals X scored by team 1 before team 2 scores their kth goal (assuming no time limit for the contest) is Negative Binomial. Try to make a connection between this situation and the usual way we define the Negative Binomial distribution (i.e., X is the number of successes before the kth failure in a sequence of independent Bernoulli trials, each having success probability p).