1. Tricycle kinematics

a. Let’s start with the analytic equations for \( \dot{x}, \dot{y}, \) and \( \dot{\theta} \) that we ended up with at the end of class. We obtained the equations:

\[
\begin{align*}
\dot{x} &= \dot{x}_R \cos \theta = \omega r \cos \alpha \cos \theta \\
\dot{y} &= \dot{x}_R \sin \theta = \omega r \cos \alpha \sin \theta \\
\dot{\theta} &= \frac{\omega r}{d} \sin \alpha \\
\end{align*}
\]

Are these linear in our purported controls \( \omega \) and \( \alpha \)? Well, \( \omega \) is looking good here, but we are clearly not linear in \( \alpha \) because it’s hiding inside a \( \cos \) in the top two equations and a \( \sin \) in the bottom one.

A kinematic system specifies a linear map from controls to configuration space velocities, and we can’t write the system above as such because it’s not linear in \( \alpha \)!

b. The fix here is to move \( \alpha \) from the controls to the configuration. We still want to control steering, however, so we must define a new control \( \dot{\alpha} \) which specifies the rate of change in the steering angle per unit time. This turns our system of three equations above into a system of four equations by adding the trivial final equation below:

\[
\begin{align*}
\dot{x} &= \dot{x}_R \cos \theta = \omega r \cos \alpha \cos \theta \\
\dot{y} &= \dot{x}_R \sin \theta = \omega r \cos \alpha \sin \theta \\
\dot{\theta} &= \frac{\omega r}{d} \sin \alpha \\
\dot{\alpha} &= \dot{\alpha} \\
\end{align*}
\]

Factoring these into matrix form, we find

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\alpha}
\end{bmatrix} =
\begin{bmatrix}
r \cos \alpha \cos \theta & 0 \\
r \cos \alpha \sin \theta & 0 \\
\frac{r}{d} \sin \alpha & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\omega \\
\dot{\alpha}
\end{bmatrix}
\]

Note that if you multiply the matrix on the left with the control vector on the right, you obtain the system of four equations above.
2. Broken tricycle

Let $R$ be the radius from the ICC to the midpoint of the hind axle, and let $\ell$ be the half the distance between the wheels. Assume (for the sake of argument) that $\alpha > 0$ so the tricycle is turning left. Let $\nu_R$ and $\nu_L$ be the velocity of the left and right wheels.

The rotational velocity of the right wheel is given by its linear velocity divided by the wheel radius, so

$$\nu_R = \frac{\dot{\theta}(R + \ell)}{r}$$

Similarly,

$$\nu_L = \frac{\dot{\theta}(R - \ell)}{r}$$

Driving the rear wheels at the same speed implies that $\nu_R = \nu_L$, which is clearly false if $\ell \neq 0$.

On the other hand, if we set $\ell = 0$, this is fine, and then we degenerate to one rear powered wheel, which is a bicycle.