What is a Robot?

- A slave (according to Teo)
- A machine that can
  - interpret surroundings
  - make decisions
  - autonomous
  - exhibit physical behavior

possible exceptions:
assembly line robots that just do the same thing over and over?
drones that are mostly controlled by humans?

Orthogonal Transformations

An \( n \times n \) matrix \( A \) is an orthogonal transform (aka unitary) if the following conditions are satisfied:

- has \( n \) unit length mutually perpendicular rows/columns
- rows/columns form an orthonormal basis of \( \mathbb{R}^n \)
- \( A^T A = A A^T = I \)
- \( A^{-1} = A^T \)

These are all mathematically equivalent, so if one of these is satisfied, all are.

Note: orthonormal means unit length, dot product is zero

Example 1: \( n = 2 \)

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Then \( x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

\( x_1 \perp x_2 \iff x_1 \cdot x_2 = 0 \)
Example 2: check that
\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\] is an orthogonal transformation

let’s check that \( AA^T = I \)

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
= \begin{bmatrix}
\cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\
\cos \theta \sin \theta - \sin \theta \cos \theta & \cos^2 \theta + \sin^2 \theta
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = I
\]

There’s no need to check that this equals \( A^T A \) because

\[(AB)^T = B^T A^T\]

The matrix used in example 2, \[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\], is known as a basic 2D rotational matrix. It has 1 degree of freedom. It is not capable of improper rotation, a transformation involving reflection.

**Orthogonal Transform Properties/Fun Facts**

- preserves dot product for orthogonal transform \( A \) and any vectors \( \vec{x} \) and \( \vec{y} \)
- closed under composition—if \( A \) and \( B \) are orthogonal transforms then so is \( A \cdot B \)

**Rotations in 3D**

For example,

\[
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

In three dimensions, there are three degrees of freedom. In an airplane, these are described as roll, pitch, and yaw.

- **roll** – rotate around the x axis, from tail to nose
- **pitch** – rotate around the y axis, from wing to wing
• **yaw** – rotate around the z axis, from ventral to dorsal

\[ R = R_x(\text{roll})R_y(\text{pitch})R_z(\text{yaw}) \]

**Angle-Axis**

How do we describe rotations in three dimensions?

• \( \vec{a} \): a unit length vector in \( \mathbb{R}^3 \) that indicates direction

• \( \alpha \): an angle to rotate around \( \vec{a} \)

Because \( \vec{a} \) won’t move when rotated around \( \vec{a} \),

\[ R\vec{a} = \vec{a} \]

which looks suspiciously similar to an eigenvector.

How do we verify a rotation matrix?
it will be an orthogonal transform with \( \det A = 1 \)

The problem with this is that even though a rotation in three dimensions has only three degrees of freedom, our description (involving a 3D vector and a 1D angle) appears to have a total of four degrees of freedom. To resolve this issue, we can use **rotation vector representation**:
given \( r \in \mathbb{R}^3 \), take \( a = \frac{r}{\|r\|}, \; \alpha = \|r\| \)

Weird cases:
• \( \vec{r} = \vec{0} \), just do the obvious thing—multiply by \( I \)
• unit quaternions (complex numbers)

**Rigid Transformations**

In this class, rigid transformations will be defined only in 2D and 3D.

• rotation + translation

• preserve distance between points

\[ T(p) = Rp + \vec{t} \]
where \( R \) is rotation and \( t \) is translation. The robot frame (with axes \( X_R \) and \( Y_R \)) can be translated and rotated relative to the world frame (with axes \( X_W \) and \( Y_W \)).

\[
P_W = R P_R + \vec{t}
\]

invert by solving for \( P_R \)

If \( T_1, T_2 \) are rigid transformations, then \( T_1 \circ T_2 = T_1(T_2(p)) \) is also a rigid transform.