9/2/14 What is a robot? It comes from Czech word for slave (R.U.R). A machine that can interpret surroundings, make decisions, is autonomous, & exhibits physical behavior. Not robots assembly line robots (?) draw?

Orthogonal Transformations

An $n \times n$ matrix $A$ is an orthogonal transformation iff (all equivalent) $\forall A$ has a mutually orthogonal unit-length rows/columns

ex. $n=2$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \& \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \text{perpendicular/orthogonal}$$

$x_1 \perp x_2 \iff x_1 \cdot x_2 = 0 = 1 \cdot 0 + 0 \cdot 1$

Continued: Properties of Orthogonal Transformations

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2 Rows/columns form an orthonormal basis of $\mathbb{R}^n$ → basis vectors dot to zero (perpendicular) & are unit-length

3 $A^T A = A A^T = I$

4 $A^{-1} = A^T$

ex. prove that $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ is an orthogonal transformation

$$A A^T = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \checkmark$$

This matrix $A$ is a rotation matrix with one degree of freedom ($\theta$). It cannot reflect.

$\Rightarrow$ Orthogonal transformations are a superset of rotation matrices.
Orthogonal Transform Properties
- preserves dot product for orthogonal transform $A$ and any vectors $x, y$: $(Ax) \cdot (Ay) = x \cdot y$
- closed under composition. If $A$ and $B$ are orthogonal transformations, so is $AB$, their product.
* important for determining robot movement

Rotations in 3D: give all possible orientations of an object, leaving some reference point fixed.

\[
A = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

- 3 degrees of freedom: roll, pitch, yaw (Euler angles)

\[R_{\text{roll}} \cdot R_{\text{pitch}} \cdot R_{\text{yaw}}\]

- angle-axis
  - can generate every possible movement in 3D
    - $a$ vector in 3D, $\alpha$ angle
    - used for direction, not magnitude
    - $Ra = a$ (a is an eigenvector that rotates about itself, with $\lambda = 1$)

- rotations: $\det A = 1$; improper rotations: $\det A = -1$
  - orthogonal transforms with determinant of 1
- rotation vector representation: Given $r \in \mathbb{R}^3$, take
  \[a = \frac{r}{||r||}, \quad \alpha = ||r|| \quad 3 \text{ numbers \rightarrow minimal transformation} \]
  * breaks for $r = 0$
- unit quaternions
Rigid Transformations: in this class, 2D or 3D

- rotation + translation
- preserves distance between points

\( T(p) = Rp + t \)

rotation + translation

- \( T \) maps from robot frame to world frame: \( P_w = R \cdot P_r + t \)

\( \Rightarrow \) to invert, solve for \( P_r \):

\( R \cdot P_r = t + P_w \)

\( P_r = R^T \cdot P_w - R^T \cdot t \)

- so the inverse of transformation given by \( R, t \) is given by \( R^T, -R^T \cdot t \)
- closed under composition