1. Vector normal to a depth map

A depth map is a function \( z = r(x, y) \) mapping points in \( \mathbb{R}^2 \) to scene depths. At some particular point \( x, y \), we can evaluate the gradient of the depth map by differentiating with respect to \( x \) and \( y \):

\[
\nabla r(x, y) = \begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix}
\]

Let’s see how this gradient relates to the normal vector of some point in the scene.

a. Why do the two vectors \( t_x = (1, 0, \frac{\partial z}{\partial x}) \) and \( t_y = (0, 1, \frac{\partial z}{\partial y}) \) form a basis of the plane tangent to the function \( r(x, y) \) at the point \( (x, y) \)? It might be useful to imagine a simple function \( r \) and to consider its cross-sections in the \( x \) and \( y \) directions.

b. Derive a formula for the normal vector \( n \) of the plane using the tangent vectors \( t_x \) and \( t_y \). Remember that \( \|n\| = 1 \) and \( n \cdot t_x = n \cdot t_y = 0 \).

c. Now, solve the inverse problem: given a normal vector \( n \) at some point \( (x, y) \), derive the gradient \( \nabla r(x, y) \) at that point.

d. Why does the formula you derived for (c) above help us reconstruct shape from shading or perform photometric shape from example? That is, explain why we would want to know the gradient, given the normal.