1. Image warping and invertible transformations

Given a digital image, and an invertible transformation \( \tilde{H} \) of the form
\[
\tilde{p}' = \tilde{H}\tilde{p}
\]
we would like to compute the warped image whereby each point \( \tilde{p} \) in the original image is transformed to its new location \( \tilde{p}' \). This type of image warping is exactly what the \texttt{cv2.warpPerspective} function does, for example.

We could envision a somewhat straightforward algorithm for performing this image warp: for each location \( \tilde{p} \) in the original image, compute the nearest pixel location of the transformed point \( \tilde{p}' \) in the warped image, and copy the color found in \( \tilde{p} \) to the warped image at location \( \tilde{p}' \).

However, the vastly preferable algorithm is to loop over the destination pixels \( \tilde{p}' \) in the warp image, and use the inverse transformation \( \tilde{H}^{-1} \) to identify the nearest pixel \( \tilde{p} \) in the source image and copy the color from that source pixel to the destination.

What is the difference between the two approaches? Why is the second one preferable?

2. Convolution vs. Correlation

Given two real-valued functions \( f \) and \( g \) of one variable, the convolution of \( f \) and \( g \) is defined as:
\[
(f * g)(x) = \int_{-\infty}^{\infty} f(x - u) g(u) \, du
\]
The correlation of \( f \) and \( g \) is defined similarly:
\[
(f \circ g)(x) = \int f(x + u) g(u) \, du
\]
Note that in the equation above, and all remaining integrals, we suppress the limits, but remember these are all definite integrals over the entire real number line.

a. Filtering confusion. Although we have defined filtering operations in class in terms of convolution, many software packages (including OpenCV) define filtering in terms of correlation instead. As we will see, it doesn’t make a huge difference.

Imagine that \( g(x) \) is a symmetric (even) function, such as a Gaussian, where
\[
g(x) = g(-x)
\]
Can you tell the difference between $f * g$ and $f \otimes g$? Why or why not?

Next, imagine that $g(x)$ is antisymmetric (odd), where

$$g(x) = -g(-x)$$

Now can you tell the difference? What is the mathematical relationship between $f * g$ and $f \otimes g$ in this case?

b. **How are convolution and correlation related?** Given any functions $f(x)$ and $g(x)$, define a function $h(x)$ such that

$$f * g = f \otimes h$$

What is $h(x)$ in terms of $g(x)$?

c. **Correlation as a dot product.** In the discrete domain, we define correlation as a sum instead of an integral:

$$(f \otimes g)(x) = \sum_u f(x + u) g(u)$$

Imagine that $f(x)$ and $g(x)$ are defined over the domain $x \in \{-2, \ldots, 2\}$. We can represent $f$ and $g$ as vectors:

$$f = \begin{bmatrix} f(-2) & f(-1) & f(0) & f(1) & f(2) \end{bmatrix}^T$$

$$g = \begin{bmatrix} g(-2) & g(-1) & g(0) & g(1) & g(2) \end{bmatrix}^T$$

Assume that $f(x) = g(x) = 0$ for $|x| > 2$. Show that the correlation $(f \otimes g)$, evaluated at $x = 0$, is equivalent to the dot product $f \cdot g$.

**3. Image derivatives and separability**

Prove that the image obtained by blurring uniformly in the $x$ and $y$ directions, and subsequently differentiating in the $x$ direction can be constructed by a separable convolution.

a. **The gaussian blur operation is separable.** Let $I(x, y)$ be a continuous function specifying image intensity at the point $(x, y)$, and let $G(x, y)$ be a Gaussian function:

$$G(x, y) = \frac{1}{Z^2} \exp \left( -\frac{x^2 + y^2}{2\sigma^2} \right)$$
where $Z^2$ is a normalizing constant that makes $G$ integrate to 1. Show that the convolution of $I$ with $G$ can be represented as convolution with two functions $g_x$ and $g_y$ that depend only on the $x$ and $y$ coordinates respectively.

Remember, the convolution $I * G$ can be expressed as the integral

$$(I * G)(x, y) = \iint I(x - u, y - v) G(u, v) \, du \, dv$$

What are the functions $g_x(x)$ and $g_y(y)$ such that $I * G = I * g_x * g_y$? It may help to define the convolution of a 2D function with a 1D function as

$$(I * g_x)(x, y) = \int I(x - u, y) g_x(u) \, du$$

and

$$(I * g_y)(x, y) = \int I(x, y - v) g_y(v) \, dv$$

b. The partial derivative of a Gaussian is separable. Let $G^{-}(x, y)$ be defined as the derivative of the Gaussian function with respect to $x$:

$$G^{-}(x, y) = \frac{\partial}{\partial x} G(x, y)$$

show that convolution $(I * G^{-})$ can also be represented as a convolution with two functions that depend only on $x$ and $y$. 