1. Root finding

Download and modify the rootfinding.py code from the course website. Then, perform the following tasks:

a. Complete the implementation of the Secant Method so the tests run correctly to find the roots of the example functions provided.

b. Ditto for Newton’s method.

c. Add a test case in the functions list to define some function \( f(x) \) for which the Bisection method outperforms Regula Falsi (i.e. uses significantly fewer iterations). Add a comment in your program indicating why Bisection works better for this function.

d. Add a test case for which the Secant Method fails to converge, but Bisection search converges. Again, add a comment in your program indicating why this happens.

Submit your modified code online.

2. Fixed Point Iteration

A fixed point of a function \( g(x) \) is a point \( x^* \) such that \( g(x^*) = x^* \). For a particular function \( g(x) \) and fixed point \( x^* \) we say that the fixed point is attractive if there exists a set of points \( x_0 \) for which the sequence

\[
x_{i+1} = g(x_i)
\]

also known as the fixed point iteration (FPI) of \( g \), converges to \( x^* \), that is

\[
\lim_{i \to \infty} x_i = x^* = g(x^*)
\]

Graphically, fixed point iteration of some function \( g(x) \) can be described as the following: starting from some initial point with \( x \)-coordinate \( x_0 \), draw a vertical line up or down until you intersect the curve \( g(x) \). Then draw a horizontal line left or right until you intersect the diagonal \( y = x \) line. The \( x \)-coordinate of the intersection becomes \( x_{i+1} \) the next value in the fixed point iteration. If you trace out the series of lines produced by this method, the resulting graph is known as a cobweb plot (http://en.wikipedia.org/wiki/Cobweb_plot).
**a.** Draw a cobweb plot of the function $g(x) = 0.5x + 1$ starting from the initial point $x_0 = 0$.

**b.** It turns out that a fixed point $x^*$ is attractive if the condition $|g'(x^*)| \leq 1$ holds. Illustrate this with a cobweb plot by showing how FPI of the function $g(x) = 2x - 1$ fails, starting from the initial point $x_0 = 2$. What goes wrong?

**c.** We can transform root finding of a function $f(x)$ into FPI by finding a suitable transformation of $f(x)$. For instance, explain why FPI on the function $g(x) = \sin(x)^{\frac{1}{3}}$ is equivalent to root finding for the function $f(x) = x^3 - \sin(x)$.

**d.** Explain how Newton’s method may be viewed as a special case of fixed point iteration. What is $g(x)$, in the case of Newton’s method?