1. Gaussian elimination with pivoting

Modify our implementation of Gaussian elimination to add pivoting. For the two systems below, have your program compute the solution \( x \) to the system \( Ax = b \). Also have your program output the corresponding upper-triangular system, before back-substitution is performed. That is, the triangular matrix \( A' \) and vector \( b' \) computed by Gaussian elimination. Submit your code as well.

\[
A = \begin{bmatrix}
0 & 3 & 2 & 1 \\
4 & 0 & 7 & 5 \\
8 & 2 & 0 & 2 \\
0 & 1 & 2 & 0 \\
\end{bmatrix}, \quad b = \begin{bmatrix}
-3 \\
2 \\
-2 \\
-5 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
0 & 2 & 6 & 1 & 2 \\
2 & 0 & 3 & 2 & 4 \\
9 & 5 & 0 & 3 & 5 \\
4 & 8 & 4 & 0 & 8 \\
1 & 0 & 0 & 4 & 0 \\
\end{bmatrix}, \quad b = \begin{bmatrix}
7 \\
-13 \\
7 \\
-4 \\
-8 \\
\end{bmatrix}
\]

2. Condition number of a matrix

For a small number \( \varepsilon > 0 \), the matrix

\[
A = \begin{bmatrix}
1 & 1 \\
1 & 1 + \varepsilon \\
\end{bmatrix}
\]

is nearly un-invertible. How much so? We can find out using the condition number of \( A \). Using the matrix norm

\[
|||A||| = \max_i \sum_j |a_{ij}|
\]

and the formula for the inverse of a \( 2 \times 2 \) matrix, compute

\[
\text{cond}(A) = |||A||| \cdot |||A^{-1}|||
\]

analytically, in terms of \( \varepsilon \).

Next, let’s examine this in practice. For \( \varepsilon = 10^{-6} \), use MATLAB to numerically compute the solution to \( Ax = b \) for \( b = [1, 1]^T \). Now observe how the solution changes as you perturb \( b \) with each of two small perturbations of \( \delta_1 = (\varepsilon, \varepsilon) \) and \( \delta_2 = (\varepsilon, -\varepsilon) \). Explain your results in terms of the condition number, and indicate why the condition number was defined in class using an inequality.