THE QUINE-MCCLUSKEY ALGORITHM
Part 2: Using a prime implicant table

The Quine-McCluskey Algorithm is yet another method for simplifying and optimizing
the Boolean expression for a binary function. Although the algorithm is functionally iden-
tical to using Karnaugh Maps, unlike K-Maps, Quine-McCluskey can be used on functions
of more than 6 variables, and more importantly, it can be automated by computers.

Quine-McCluskey is a two-step process: first, we identify a sufficient set of prime
implicants, and then we use a prime implicant table to reduce that set to the smallest
possible number of implicants. This worksheet describes the second step.

DEFINITIONS

A prime implicant for a Boolean function is a product term that implies the function; that
is, if the prime implicant is equal to 1, then the function is equal to 1. For instance, the
function
\[ F(x, y, z) = xy'z' \]
has two prime implicants: \( xy \) and \( x'y'z' \).

A prime implicant is said to cover a minterm when it contains a subset of the variables
constituting the minterm. The prime implicant \( xy \) covers two minterms: \( xyz \) and \( xyz' \).

IDENTIFYING COVERED MINTERMS

Let’s look at what minterms of the function \( F(w, x, y, z) \) are covered by the prime
implicant \( xy' \). Since only two variables are specified, there will be four minterms covered.
First, we can distribute \((w' + w)\) into the product in order to yield two product terms:
\[ xy' = 1 \cdot (xy') = (w' + w) \cdot xy' = w'xy' + wxy' \]
Next we can distribute \((z' + z)\) into both product terms in order to yield the four minterms
covered by \( xy' \):
\[ w'xy' + wxy' = w'xy'(z' + z) + wxy'(z' + z) = w'xy'z' + w'xy'z + wxy'z' + wxy'z = \Sigma(4, 5, 12, 13) \]
EXAMPLE

We begin with a function written as a sum of prime implicants:

\[ F(A, B, C, D) = A'C'D + A'BD + A'BC + B'C' + B'D' + CD' \]

Now we need to see which minterms are covered by these prime implicants. First, let’s look at the three-variable terms. They each cover two minterms:

- \( A'C'D = A'B'C'D + A'BCD = \Sigma(1, 5) \)
- \( A'BD = A'BCD + A'BC'D = \Sigma(5, 7) \)
- \( A'BC = A'BC'D + A'BCD = \Sigma(6, 7) \)

Next, let’s look at the two-variable terms. They each cover four minterms:

- \( B'C' = A'B'C' + AB'C' = A'B'C'D + A'BC'D + AB'C'D + AB'C'D = \Sigma(0, 1, 8, 9) \)
- \( B'D' = A'B'D' + AB'D' = A'B'C'D' + A'BC'D' + AB'C'D' + ABCD = \Sigma(0, 2, 8, 10) \)
- \( CD' = A'CD' + ACD' = A'B'CD' + A'BCD' + AB'CD' + ABCD = \Sigma(2, 6, 10, 14) \)

If we gather up all the minterms that are covered, we see that

\[ F(A, B, C, D) = \Sigma(1, 5) + \Sigma(5, 7) + \Sigma(6, 7) + \Sigma(0, 1, 8, 9) + \Sigma(0, 2, 8, 10) + \Sigma(2, 6, 10, 14) \]

Any combination of prime implicants which covers all of the minterms in the standard sum of products above is a valid expression of our function \( F(A, B, C, D) \). We will now use a prime implicant table to see if we can get rid of any product terms.
We construct the prime implicant table by writing out the prime implicants and the minterms they cover on the left hand side. The columns of the right hand side are all the minterms covered by the function. In each row, we place an “X” to indicate which minterms are covered by the prime implicant in that row.

<table>
<thead>
<tr>
<th>1, 5</th>
<th>$A'C'D$</th>
<th>X</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>5, 7</td>
<td>$A'BD$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6, 7</td>
<td>$A'BC$</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>0, 1, 8, 9</td>
<td>$B'C'$</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>0, 2, 8, 10</td>
<td>$BD'$</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>2, 6, 10, 14</td>
<td>$CD'$</td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>

To find our minimal set of prime implicants, we follow these rules:

- If a particular minterm is only covered by one prime implicant, we choose that prime implicant, and cross out all of the minterms that it covers.
- If there is no minterm remaining that is only covered by one prime implicant, we choose one that covers the greatest number of remaining minterms, and cross out all of the minterms that it covers.
- If all prime implicants cover an equal number of remaining minterms, we can choose one arbitrarily, and cross out all of the minterms that it covers.
- When there are no minterms remaining, we are finished, and the prime implicants we have chosen form a minimal expression of our function.

Looking at the table above, we can see that the only prime implicant covering minterm 9 is $B'C'$, so we choose it and cross out the minterms that it covers.

<table>
<thead>
<tr>
<th>1, 5</th>
<th>$A'C'D$</th>
<th>X</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>5, 7</td>
<td>$A'BD$</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>6, 7</td>
<td>$A'BC$</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>0, 1, 8, 9</td>
<td>$B'C'$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0, 2, 8, 10</td>
<td>$BD'$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2, 6, 10, 14</td>
<td>$CD'$</td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>
Next, we see that the only prime implicant which covers minterm 14 is $CD'$, so we choose it and cross out all of the minterms which it covers:

\[
\begin{array}{c|cccccccccc}
\text{minterms} & 0 & 1 & 2 & 5 & 6 & 7 & 8 & 9 & 10 & 14 \\
\hline
1, 5 & A'CD & X & & & & & & & & \\
5, 7 & A'BD & & X & & & & & & & \\
6, 7 & A'BC & & & X & & & & & & \\
0, 1, 8, 9 & B'C' & & & & & & & & & \\
0, 2, 8, 10 & B'D' & & & & & & & & & \\
2, 6, 10, 14 & CD' & & & & & & & & & \\
\end{array}
\]

Now there are only two minterms left to cover. The first rule doesn’t apply anymore because there are no minterms left that are only covered by one prime implicant. So, using the second rule, we choose the prime implicant $A'BD$ because it covers both remaining minterms.

\[
\begin{array}{c|cccccccccc}
\text{minterms} & 0 & 1 & 2 & 5 & 6 & 7 & 8 & 9 & 10 & 14 \\
\hline
1, 5 & A'CD & X & & & & & & & & \\
5, 7 & A'BD & & X & & & & & & & \\
6, 7 & A'BC & & & X & & & & & & \\
0, 1, 8, 9 & B'C' & & & & & & & & & \\
0, 2, 8, 10 & B'D' & & & & & & & & & \\
2, 6, 10, 14 & CD' & & & & & & & & & \\
\end{array}
\]

We can now verify that the prime implicants we have selected cover all of the original minterms covered by the function:

\[
A'BD + B'C' + CD' = \Sigma(5, 7) + \Sigma(0, 1, 8, 9) + \Sigma(2, 6, 10, 14)
\]
\[
= \Sigma(0, 1, 5, 6, 7, 8, 9, 10, 14)
\]
\[
= F(A, B, C, D)
\]

And so we have identified a minimal set of prime implicants to cover the function. In practice, identifying which minterms are covered by the prime implicants is performed during the first step of Quine-McCluskey, so you can typically skip straight to constructing the prime implicant table. Nevertheless, it’s a good skill to have if you need to simplify an arbitrary sum of products!
EXERCISES

1. Identify which minterms are covered by each prime implicant of the function

\[ F(w, x, y, z) = w'xz + xyz + wx' + wy \]

2. Write \( F(w, x, y, z) \) as a sum of standard products.

3. Construct a prime implicant table and use it to reduce the sum of prime implicants above to a minimal set.