1. Ergodic Averaging and Configurations of Integers

My research uses methods of dynamical systems to study questions that arise related to combinatorial number theory. In 1975, Szemerédi proved the long-standing conjecture of Erdös and Turán that every subset of the integers with positive upper density[1] contains arbitrarily long arithmetic progressions. Soon after, Furstenberg provided an alternate proof of Szemerédi's Theorem, using techniques of ergodic theory. He developed a Correspondence Principle which establishes an equivalence between configurations within the integers, such as arithmetic progressions, and recurrence properties of measure preserving systems. My research primarily focuses on understanding properties of such measure preserving systems.

A **measure preserving system** is a quadruple $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a probability measure space, and $T\colon X \to X$ is an invertible, measurable, measure preserving transformation, i.e. for any $A \in \mathcal{B}$, $T^{-1}A \in \mathcal{B}$, and $\mu(T^{-1}A) = \mu(A)$.

Furstenberg's Correspondence Principle: Let $\overline{d}(E)$ be the upper density of $E \subseteq \mathbb{Z}$, and suppose $\overline{d}(E) > 0$. We can find a measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) = \overline{d}(E)$ such that

$$\mu(T^{-n_1}A \cap T^{-n_2}A \cap \ldots \cap T^{-n_k}A) \leq \overline{d}((E+n_1) \cap (E+n_2) \cap \ldots \cap (E+n_k))$$

for any $k \in \mathbb{N}$ and any $n_1, \ldots, n_k \in \mathbb{Z}$.

To show E contains an arithmetic progression $x, x+n, ..., x+kn$ with $n \neq 0$, it is sufficient to prove for any measure preserving system, $\mu(A \cap T^{-n}A \cap \ldots \cap T^{-kn}A)) > 0$ for some $n \neq 0$.

Furstenberg completed his alternate proof for Szemerédi's theorem by proving the following multiple recurrence theorem: Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. For every $k \in \mathbb{N}$ and every $A \in \mathcal{B}$ with $\mu(A) > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0.$$

In particular, this means that there are infinitely many n with $\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0$.

Ergodic theorists have since proved similar combinatorial theorems using this strategy, such as the following multidimensional polynomial version of Szemerédi's theorem by Bergelson and Leibman [BL]. Let $(X, \mathcal{B}, \mu)$ be a probability space, let $T_1, \ldots, T_l\colon X \to X$ be commuting invertible measure preserving transformations, let $p_i\colon \mathbb{Z} \to \mathbb{Z}$ be polynomials satisfying $p_i(0) = 0$ for all $1 \leq i \leq l$, and let $A \in \mathcal{B}$ with $\mu(A) > 0$. Then

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(T_1^{-p_1(n)}A \cap \ldots \cap T_l^{-p_l(n)}A) > 0. \tag{1}$$

For example, using Furstenberg's Correspondence Principle, it follows that every set of positive upper density contains polynomial configurations of the type $\{x+n, x+n^2, x+n^3\}$.

In proving these combinatorial results, there has been much work into understanding the underlying structures and convergence behavior of the above averages in (1). We say a measure preserving transformation $T\colon X \to X$ is **ergodic** if for any measurable set A, $T^{-1}A = A$ implies that $\mu(A) = 0$, or $\mu(A) = 1$. We say a group of commuting invertible measure preserving transformations $T_1, \ldots, T_l\colon X \to X$ is called **totally ergodic** if each $T_1^{a_1} \ldots T_l^{a_l}$ is ergodic for all

[1] The upper density of a set $E \subseteq \mathbb{Z}$, is defined to be $\limsup_{N\to\infty} \frac{|E \cap \{1,\ldots,N\}|}{N}$.

$(a_1, \ldots, a_l) \neq (0, \ldots, 0)$. When our group of transformations is totally ergodic, I showed that the averages in (1) indeed converge. More generally, I show convergence for a group of L^∞ functions, instead of a characteristic function of a single set.

Theorem. [J] *Let $(X, \mathcal{B}, \mu)$ be a probability space, let $T_1, \ldots, T_l$ be totally ergodic group of commuting invertible measure preserving transformations of X, and let $p_i \colon \mathbb{Z} \to \mathbb{Z}$ be polynomials for $1 \leq i \leq l$. Then for any $f_1, \ldots, f_r \in L^\infty(\mu)$, the averages*

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T_1^{p_1(n)}x)f_2(T_2^{p_2(n)}x)\ldots f_l(T_l^{p_l(n)}x) \tag{2}$$

converge in $L^2(\mu)$ as $N \to \infty$.

Without the assumption of totally ergodicity, whether the commuting polynomial averages (2) converges still remains an open question. For a single transformation, convergence was shown by Host and Kra [HK2], and Leibman [Le2]. Recently, Tao [Ta] showed that for any group of commuting transformations $T_1, \ldots, T_l$ and any functions $f_1, \ldots, f_n \in L^\infty$, the linear commuting averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T_1^n x)\ldots f_l(T_l^n x)$$

converge in $L^2(\mu)$. Previously, Frantzikinakis and Kra [FrK] proved convergence for linear polynomials of commuting transformations assuming each T_i, and each $T_iT_j^{-1}$ is ergodic.

The first step in the proof is to reduce to a simpler system, called a characteristic factor, for which the averages do not change when each function is replaced by its projection. If our system is totally ergodic, I show that the factors developed by Host and Kra in [HK1] are characteristic factors for the averages (1). These factors are isomorphic to an inverse limit of k-step nilsystems, where convergence was shown by Leibman [Le1]. For the general commuting case, one might hope that the Host-Kra factors would also be characteristic. Simple examples show that even for linear polynomials, this is not the case. It is clear that new methods are needed to prove convergence for the general commuting polynomial averages.

In light of these convergence results, we are interested in finding other sequences with similar convergence results. We say a sequence of positive integers $\{s_1, s_2, \ldots\}$ is **good for mean convergence** if for all measure preserving systems $(X, \mathcal{B}, \mu, T)$ and all $f \in L^\infty(\mu)$, the averages $\frac{1}{N}\sum_{n=0}^{N-1} f(T^{s_n}x)$ converges in $L^2(\mu)$. While much is unknown about sequences that are good for mean convergence, examples include polynomial sequences $\{p(1), p(2), \ldots\}$ where $p(n)$ is any integer valued polynomial, [HK2] and [Le2].

Given any sequence $\{s_1, s_2, \ldots\}$ that is good for mean convergence, we ask if its sequence of squares $\{s_1^2, s_2^2, \ldots\}$ is also good for mean convergence. Surprisingly, the answer is no. In joint work with Nikos Frantzikinakis, Emmanuel Lesigne, and Mate Wierdl [FJLW], we proved the following generalization:

Theorem. [FJLW] *Given any set of "bad" exponents $K \subseteq \mathbb{N}$, there is a sequence $\{s_1, s_2, \ldots\}$ such that $S^k = \{s_1^k < s_2^k < \ldots\}$ is good for mean convergence if and only if k is not in K.*

2. Rank one $\mathbb{Z}^d$-actions

More recently, I have been studying rank one $\mathbb{Z}^2$-actions of measure preserving systems. Kakutani and Rohlin developed a theory describing the structure of ergodic measure preserving systems using a sequence of finite towers. More precisely, for each $\varepsilon > 0$, we can find a set $A \in \mathcal{B}$ and $n \in \mathbb{N}$ such that $A, TA, \ldots, T^n A$ are pairwise disjoint, and $\mu(\bigcup_{k=0}^n T^k A) > 1 - \varepsilon$. We call the partition $\tau = \{A, TA, \ldots, T^{n-1}A, (\bigcup_{k=0}^n T^k A)^c\}$ a tower with levels $A, TA, \ldots, T^n A$, and error set $(\bigcup_{k=0}^n T^k A)^c$. A transformation T is called **rank one** if there is a sequence of towers τ_i where the levels of τ_i, generate the σ-algebra $\mathcal{B}$. It is known that all rank one transformations are loosely Bernoulli.

We generalize this notion of rank one to systems, called $\mathbb{Z}^2$-actions, with two commuting measure preserving transformations S and T, by constructing 2-dimensional towers whose levels at each step i are $S^{c_1}T^{c_2}(A)$, where (c_1, c_2) come from some rectangular lattice $\mathcal{R}_i = [0, m_i] \times [0, n_i]$. We say a sequence of rectangles $\mathcal{R}_i = [0, m_i] \times [0, n_i]$ has bounded eccentricity if there are bounds b_1, b_2 such that $0 < b_1 < \frac{m_i}{n_i} < b_2$ for all integers i. It was shown by A. Johnson and Sahin [JS] that rank one $\mathbb{Z}^2$-actions whose sequence of rectangles $\mathcal{R}_i$ have bounded eccentricity are loosely Bernoulli. It remains an open whether general rank one $\mathbb{Z}^2$ actions are also loosely Bernuolli. I am currently involved in joint work with Aimee Johnson and Anne McCarthy, to determine how more general tower shapes impact whether a $\mathbb{Z}^2$-action is loosely Bernoulli.

3. Future Plans

Loosely Bernoulli systems are defined using a metric related to both the Hamming distance and permutations of finite integer lattices. I am interested in finding a useful characterization of when certain 2-dimensional arrays are close in this metric. Using a brief introduction to symbolic dynamics, this can be turned into a nice research project for an undergraduate student.

I am also interested in other open questions related to ergodic averages, such as polynomial generalizations of Khintchine recurrence for commuting transformations. It is unknown whether for totally ergodic systems if the set of $\{n\colon \mu(A \cap T^{-n}A \cap S^{-n^2}A) > 0\}$ is syndetic, i.e., has bounded gaps. To prove such a statement, more needs to be known about specific characteristic factors for the corresponding ergodic averages. Understanding these characteristic factors has the potential to answer other interesting questions in combinatorial ergodic theory.

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