E58: Control Theory

Midterm

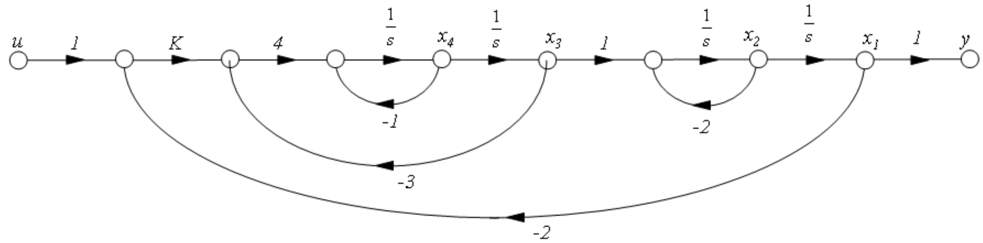
Due 3/7 at 12:00pm

Read this before you start.

1. Remember to write your name on top of this page and any additional pages that are not stapled to this page.
2. You must complete the exam in 120 minutes.
3. All work must be clearly shown for points to be awarded.
4. Clearly state all your assumptions.
5. You can ONLY refer to your book, notes, and handouts provided in class.
6. You can ONLY use Matlab, Mathematic, or Maple to help you with the calculations.
7. You must complete the exam on your own.
8. You may NOT discuss the exam until everyone has turned in their exams on Friday 3/7 at 12:00pm.
9. Make sure you turn in your completed exam to Ms. Holly Castleman no later than Friday 3/7 at 12:00pm. She will have a sign–in sheet for you. Please make sure you sign–in when you turn in your exam, otherwise, your exam will be not be considered!

Problem 1: 25 points

For the following system



(a) Write the state equations for this system;

(b) Determine the transfer function; and

(c) Determine the range of K where the system is stable.

(a) The state equations are given by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2x_2 + x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -8Kx_1 - 12x_3 - x_4 + 4Ku \\
y &= = x_1
\end{aligned}$$

which results in the

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8K & 0 & -12 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4K \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \mathbf{D} = \mathbf{0}.$$

(b) Then the transfer function for the system is given by

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

Using Matlab's Symbolic Math Toolbox, $(s\mathbf{I} - \mathbf{A})^{-1}$ is determined by

>> `sym s K;`
>> $A = [0\,1\,0\,0; 0\ -2\,1\,0; 0\,0\,0\,1; -8K\,0\ -12\ -1]$;
>> $inv(s * eye(4) - A)$

$$\frac{1}{(s^4 + 3s^3 + 14s^2 + 24s + 8K)} \begin{bmatrix} (s+2)(s^2+s+12) & (s^2+s+12) & (s+1) & 1 \\ -8K & s(s^2+s+12) & s(s+1) & s \\ -8K(s+2) & -8K & s(s+2)(s+1) & (s+2)s \\ -8Ks(s+2) & -8Ks & -4(3s^2+6s+2K) & s^2(s+2) \end{bmatrix}$$

As such, the transfer function is given by

$$\begin{aligned} T(s) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} (s\mathbf{I} - \mathbf{A})^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4K \end{bmatrix} \\ &= \frac{4K}{(s^4 + 3s^3 + 14s^2 + 24s + 8K)} \end{aligned}$$

(c) To find the range of K where the system is stable, we construct the following Routh table: Thus the system is table for $0 < K < 6$.

s^4	1	14	$8K$
s^3	3	24	0
s^2	$-(24 - 3(14))/3 = 6$	$-(0 - 3(8K))/3 = 8K$	0
s^1	$-(3(8K) - 24(6))6 = 24 - 4K$	0	0
s^0	$8K$	0	0

Problem 2: 25 points

Let A be an $n \times n$ constant matrix. We can define the matrix exponential, e^{At} as an $n \times n$ matrix, such that e^{At} is the solution to the matrix differential equation

$$\frac{d}{dt} F(t) = AF(t)$$

where $F(0) = I = n \times n$ identity matrix. In other words, e^{At} satisfies the above matrix differential equation and $e^{A0} = I$.

(a) Use the facts given so far and the property of the Laplace Transforms

$$\mathcal{L}\{\dot{F}(t)\} = s\mathcal{L}\{F(t)\} - F(0)$$

to show that

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

(b) Even if you don't prove the previous relation, use it to find e^{At} (in close form) if

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

(c) Lastly, consider the following system

$$\begin{aligned} \dot{x}_1 &= x_1 - Kx_1^3 + x_2 \\ \dot{x}_2 &= 3Kx_1 - x_2. \end{aligned}$$

Find all the equilibrium points of the system. Using linearization, for what range of K is the system stable? K can be positive or negative.

(a) Let $\hat{F}(s)$ denote the Laplace transform of $F(t)$, then taking the Laplace transform of both sides of the equation $dF(t)/dt = AF(t)$, we get

$$\begin{aligned} s\hat{F}(s) - F(0) &= A\hat{F}(s) \\ (sI - A)F(s) &= F(0) = I \\ F(s) &= (sI - A)^{-1} \end{aligned}$$

Since we know that the solution is $F(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$ and we are told that the solution to the differential equation is e^{At}, we can conclude (or more formally, due to the uniqueness of the solution of the differential equation with initial conditions $F(0) = I$) that $\mathcal{L}^{-1}\left[(sI - A)^{-1}\right] = e^{At}$.

(b) Then given A, e^{At} is determined as follows:

$$\begin{aligned} (sI - A)^{-1} &= \begin{bmatrix} s-1 & -2 \\ 0 & s-1 \end{bmatrix}^{-1} \\ &= \frac{1}{(s-1)^2}\begin{bmatrix} s-1 & 2 \\ 0 & s-1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s-1} & \frac{2}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix} \end{aligned}$$

As such,

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] \\ &= \begin{bmatrix} e^t & 2te^t \\ 0 & e^t \end{bmatrix}. \end{aligned}$$

(c) To find the equilibrium points of the system, set $\dot{x}_1 = \dot{x}_2 = 0$

$$\begin{aligned} 0 &= x_1 - Kx_1^3 + x_2 \\ 0 &= 3Kx_1 - x_2 \end{aligned}$$

solving for x_1 and x_2 we get $(0,0)$, $(\pm\sqrt{(3K+1)/K}, \pm 3K\sqrt{(3K+1)/K})$. To determine the stability of the system, we linearize about these equilibrium points and find the ranges of K such that the eigenvalues of the linearized system are all negative. Linearizing the system, we get

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 1 - 3Kx_1^2 & 1 \\ 3K & -1 \end{bmatrix}_{(x=x_0)} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

For $(0,0)$, the above matrix becomes

$$\begin{bmatrix} 1 & 1 \\ 3K & -1 \end{bmatrix}$$

with eigenvalues $\lambda_{1,2} = \pm\sqrt{1-3K}$. Thus, the system is at best marginally stable, *i.e.* the eigenvalues are purely imaginary.

For $(\pm\sqrt{(3K+1)/K}, \pm 3K\sqrt{(3K+1)/K})$, the matrix becomes

$$\begin{bmatrix} s + 9K & 1 \\ 3K & -1 \end{bmatrix}$$

with engivalues $\lambda_{1,2} = -(9K+1)/2 \pm \sqrt{(9K+1)^2 - 24K}/2$. Thus, the system is stable if $K < 1/9$.

Problem 3: 25 points

For the following system

$$G(s) = \frac{1}{s^3 + 10s^2 + 3s + 10}$$

(a) Design a PD compensator, denoted as $G_d(s)$, to yield a closed–loop response with 10% overshoot and a settling time of 1.5 seconds.

(b) Add a PI compensator to your compensated system in (a), *i.e.* the plant is now $G_d(s)G(s)$. Design your PI compensator to yield a closed–loop response that best matches the specifications in (a). What are the trade–offs?

Clearly outline your design methodology. Include the root locus plots for both the uncompensated and compensated systems in both (a) and (b). List the closed–loop poles for each compensated system and include the closed–loop step response of each compensated system.

Extra page for Problem 3.

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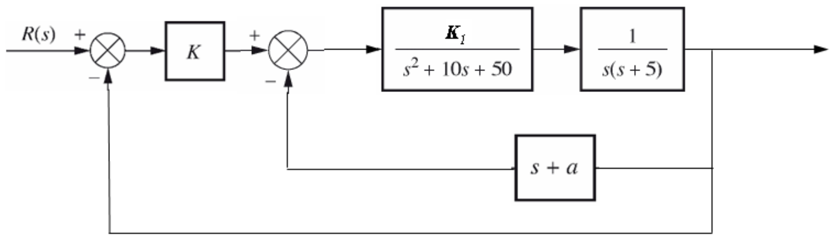
Problem 4: 25 points

For the following system



(a) Design the values of K_1 and a to yield a closed–loop response with 5% overshoot and a settling time of 1 second for the minor loop.

(b) Design the value of K to yield a major–loop response with 10% overshoot for a step input.

(c) Plot your step input response for the closed–loop system. Discuss your results.

Once again clearly outline your design methodology. Include all root locus plots and list all closed–loop poles.

Extra page for Problem 4.

Extra page for Problem 4.

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Problem	Grade	Max
1		25
2		25
3		25
4		25
Total		100