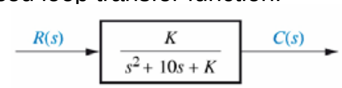
Root Locus Recap

- Root Locus: a method of presenting graphical information about a system's behavior when the controller is working
  - Common tool for design of closed loop systems
  - Allows us to sketch out system behavior for a range of K

Example 1

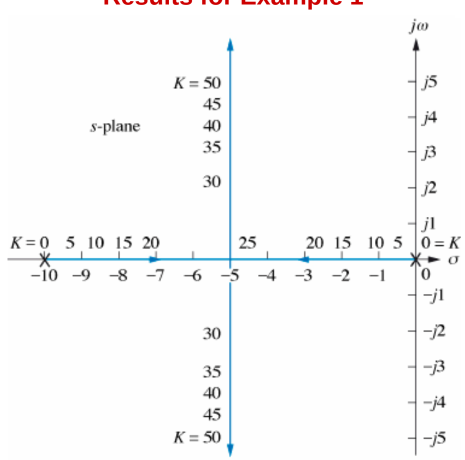
Given the closed loop transfer function:



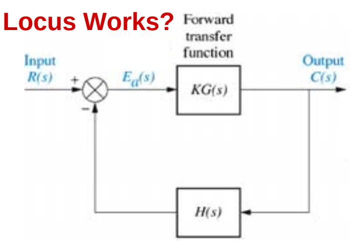
Look at poles for different values of K

K	Pole 1	Pole 2
0	−10	0
5		
10		
15		
20		
25		
30		
35		
40		
45		
50		

Results for Example 1



Why the Root Locus Works?



- Closed loop transfer function

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

- Poles occur when

$$KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ$$

  - Must consider both magnitude and angle

Rules for the Root Locus

- Number of branches = close loop poles
  - Branch ◷ path that one pole traverses as the gain is varied
- Root Locus is symmetric about the real axis
- The root locus segments lie on the real axis to the left of an odd number of open loop poles and zeros
- The root locus begins (0 gain) at the poles and ends (∞ gain) at the zeros (finite and infinite) of G(s)H(s)
- Asymptotes

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\sharp \text{finite poles} - \sharp \text{finite zeros}}$$

$$\theta_a = \frac{(2k+1)\pi}{\sharp \text{finite poles} - \sharp \text{finite zeros}}$$

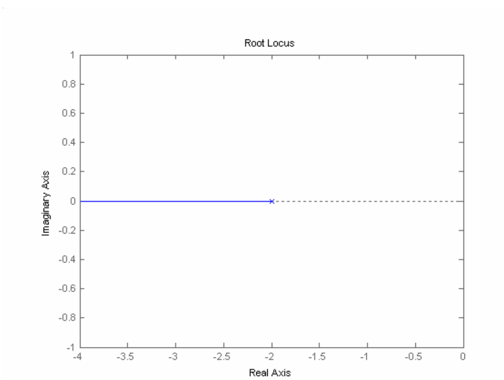
- Break-out & Break-in Points $\frac{d\,[G(s)H(s)]}{ds} = 0$

Low Order Loci

- Use only the first few rules
  - Use the rules in order
- Practice sketching loci to gain proficiency
- The following are some examples of low order loci

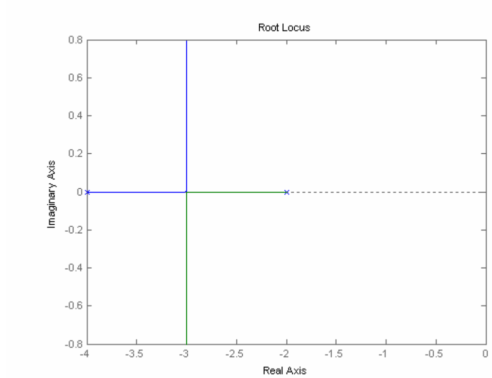
One Pole

- 1/(s+2) w/ pole = -2



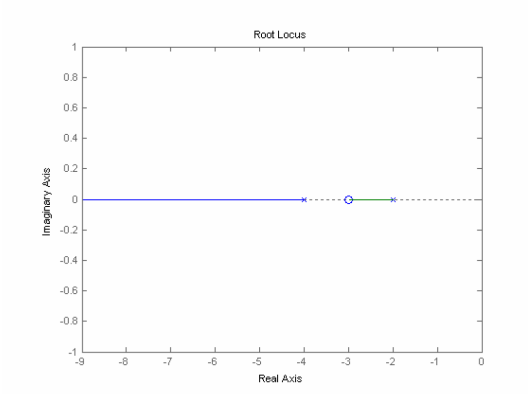
Two Poles

$1/(s^2 + 6s + 8)$ with poles at -2, -4



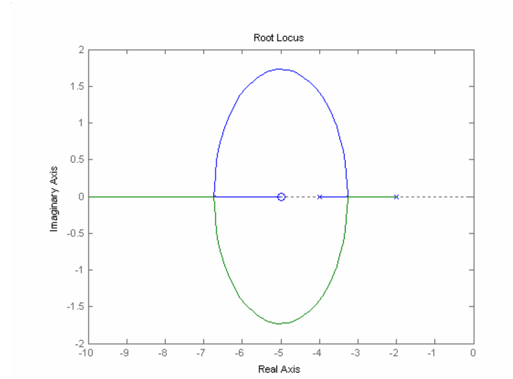
One Zero, Two Poles

$(s+3)/(s^2 + 6s + 8)$ with one zero at -3 and poles at -2, -4



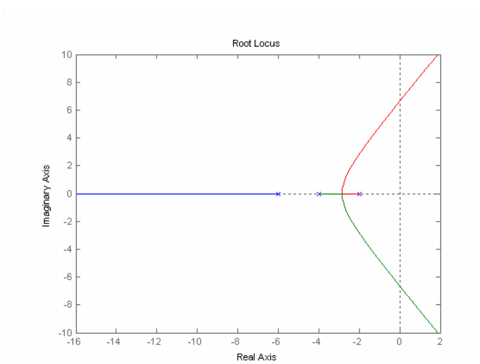
Zero Outside Two Poles

$(s+5)/(s^2 + 6s + 8)$ with one zero at -5 and poles at -2, -4



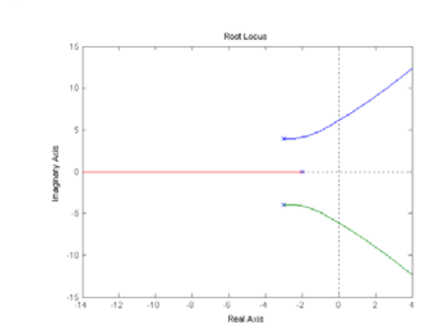
Three Poles

$1/(s^3 + 12s^2 + 44s + 48)$ with poles at -2, -4, -6



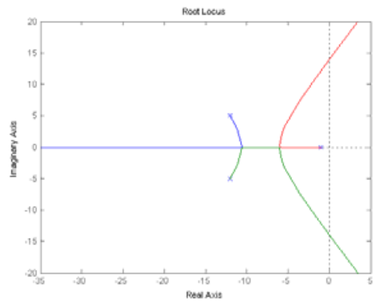
No Breakaway Points

$1/(s^3 + 8s^2 + 37s + 50)$ with poles at -2, $-3 \pm j4$



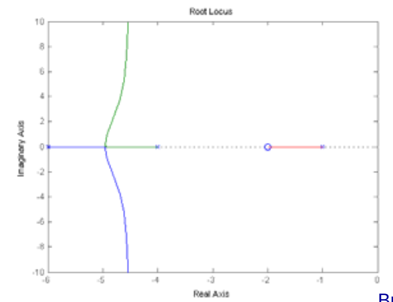
With Breakaway Points

$1/(s^3 + 25s^2 + 193s + 169)$ with poles at -1, -12 ± j5



One Zero, Three Poles

$(s+2)/(s^3 + 11s^2 + 34s + 24)$ with one zero at -2 and poles at -1, -4, -6



Asymptotes

$$\sigma_a = \frac{(-1-4-6)-(-2)}{3-1} = -4.5$$

Breakaway Points

$$\mathrm{root}\left(2 \cdot s^3 + 17 \cdot s^2 + 44 \cdot s + 44, s\right) = -4.957$$

A Few More Details on the Root Locus

- jω-axis crossings
  - Obtain this via Routh-Hurwitz criterion
  - Force a row of zeros in the Routh table to get this gain
  - Once we have the gain, we can solve for s
- Consider $T(s) = \dfrac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$

Transient Response Design via Gain Adjustment

- Design Procedure for Higher Order Systems
  1. Sketch the RL
  2. Assume 2nd order system w/ no zeros
  3. Find K to meet transient response specs
  4. Verify positions of higher order poles to make sure assumptions are valid
  5. If assumptions do not hold, simulate system numerically

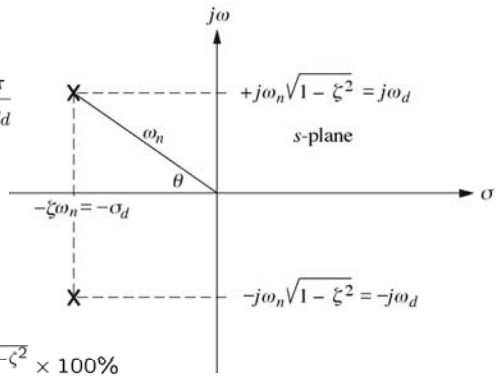
Recall From Chapter 4

$$G(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \qquad s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

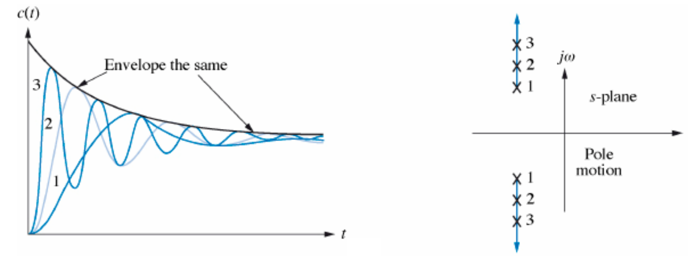
$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%$$



Peak Time

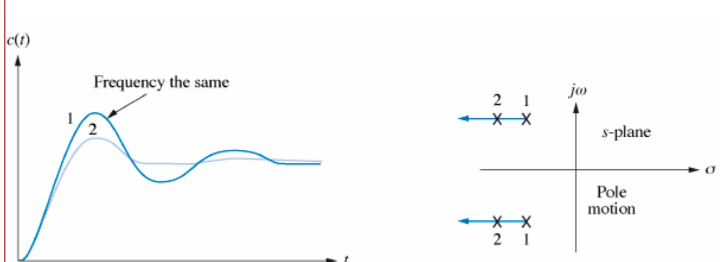
- As ω_d increases, T_p decreases



$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

Settling Time

- As $-\sigma_d$ increase, T_s decreases



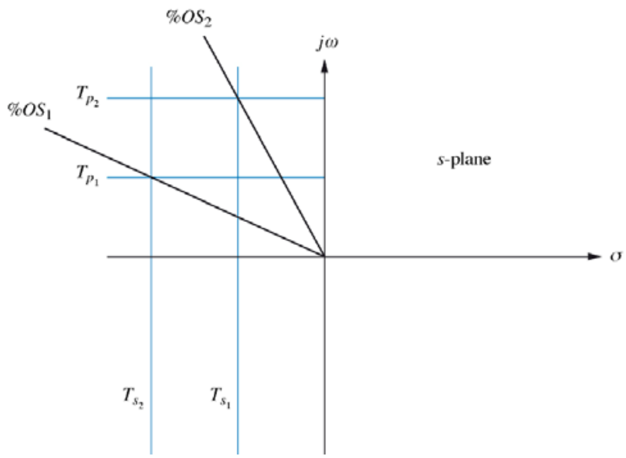
$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

Overshoot

- Let θ be the angle made by $-\sigma_d + j\omega_d$
- $\sigma_d = \cos\theta$

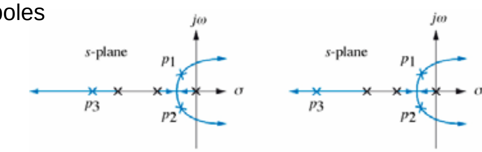
$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%$$

In Summary

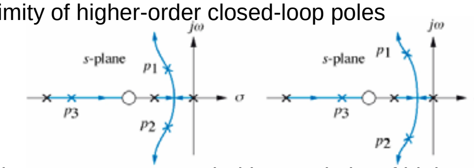


Conditions for Second-Order Approximations

- Higher order poles are much farther to the left of the dominant 2nd order poles

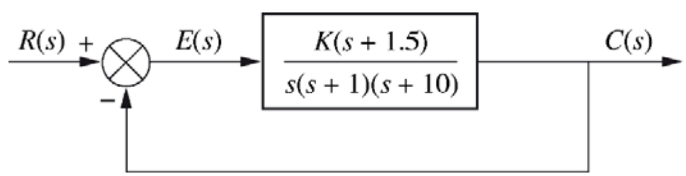


- Closed-loop zeros near closed-loop 2nd order poles are canceled by proximity of higher-order closed-loop poles



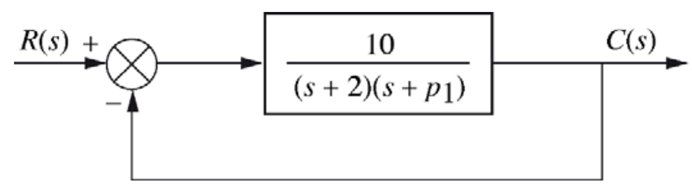
- Closed-loop zeros not canceled by proximity of higher-order closed-loop poles are far from closed-loop 2nd order poles

Example



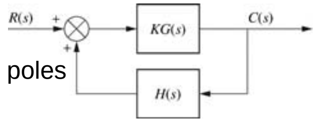
- Design value of K such that we get 1.52% overshoot.
- Estimate settling time, peak time, and steady-state error.

Generalized Root Locus



- What about this system?
- Can we determine the behavior of the system for a range of p_1 values?
- Answer: Yes!

Root Locus for Positive Feedback Systems



1. Number of branches = # of closed-loop poles
2. Symmetry
3. Real Axis Segments: to the left of even (vs. odd) # of finite poles and/or zeros
4. Starting & ending points
5. Asymptotes & intercepts

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\sharp\text{finite poles} - \sharp\text{finite zeros}}$$

$$\theta_a = \frac{2k\pi}{\sharp\text{finite poles} - \sharp\text{finite zeros}}$$

6. Break-out & break-in points $\frac{d\,[G(s)H(s)]}{ds} = 0$

Example