

Re-cap

- So far we are able to
 1. Model dynamical systems
 2. Use block diagrams/signal flow graphs to graphically represent our systems
 3. Simplifying our block diagrams/signal flow graphs
 4. Obtaining solutions to LTI SISO systems via the Laplace Transform
 5. Obtaining solutions to LTI MIMO systems using state-space representation
 6. Characterize our systems performance: 1st and 2nd order systems
 - Transient response
 - Steady-state response



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Yet Another Analysis & Design Tool

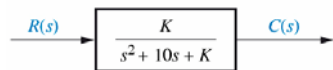
- Root Locus: a method of presenting graphical information about a system's behavior when the controller is working
 - Common tool for design of closed loop systems
- Why Root Locus?
 - What happens when the gain of the controller changes?
 - Will the system be stable?
 - Will the response change?



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Example 1

Given the closed loop transfer function:



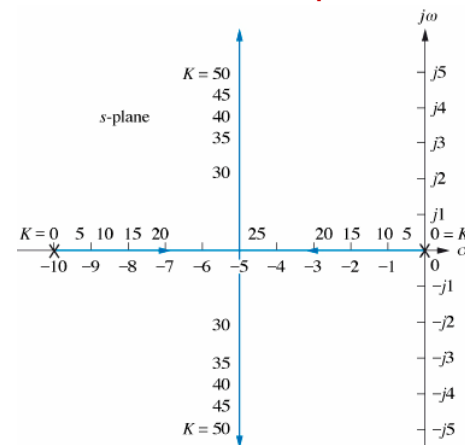
Look at poles for different values of K

K	Pole 1	Pole 2
0	-10	0
5
10
15
20
25
30
35
40
45
50



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Results for Example 1



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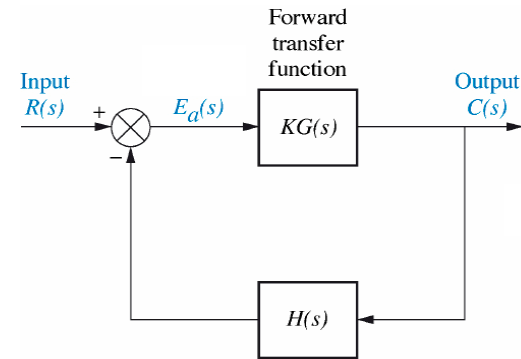
Complex Number Review

- Complex number $s = \sigma + j \omega$
 - Represented by a **vector** in Cartesian coordinates
- Given $F(s) = s+a$
 - If $s = \sigma + j \omega$, then $F(s)$ is also complex
- Given $s = \sigma + j \omega$, the complex has both
 - Magnitude, $M = \sqrt{\sigma^2 + \omega^2}$
 - Angle, $\theta = \tan^{-1}(\omega/\sigma)$
- In polar coordinates, we express $s = \sigma + j \omega$ as $M\angle\theta$



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Closed Loop System



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Transfer Function

- Closed loop transfer function

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

- Poles occur when

$$KG(s)H(s) = -1 = 1\angle(2k + 1)180^\circ$$



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Two Parts to Consider

- The magnitude of the characteristic equation

$$|KG(s)H(s)| = 1 \quad K = \frac{1}{|G(s)H(s)|}$$

- The angle of the characteristic equation

$$\angle KG(s)H(s) = \angle(2k + 1)180^\circ$$



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Magnitude and Angles of Characteristic Equations

Given a function

$$F(s) = \frac{\prod_{i=1}^m (s + z_i)}{\prod_{j=1}^n (s + p_j)}$$

The magnitude of F(s) is given by

$$M = |F(s)| = \frac{\prod_{i=1}^m |(s + z_i)|}{\prod_{j=1}^n |(s + p_j)|}$$

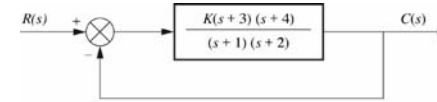
The angle of F(s) is given by

$$\theta = \angle F(s) = \sum_{i=1}^m \angle(s + z_i) - \sum_{j=1}^n \angle(s + p_j)$$



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Example 2



The closed loop system transfer function is:

$$T(s) = \frac{K(s+3)(s+4)}{(1+K)s^2 + (3+7K)s + (2+12K)}$$

With KG(s)H(s) given by $\frac{K(s+3)(s+4)}{(s+1)(s+2)}$

Consider $s = -2 + j3$

The point $s = -2 + j3$
IS NOT on the root
locus!

All points of on the root locus result in $\angle KG(s)H(s) = (2k+1)180^\circ$



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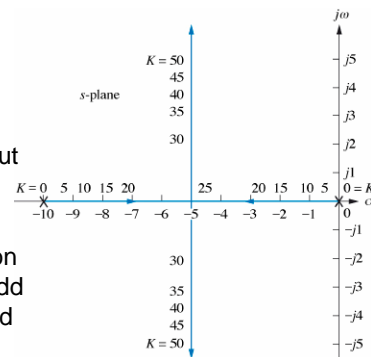
Rules for the Root Locus

• Number of branches = close loop poles

- Branch \odot path that one pole traverses as the gain is varied

• Root Locus is symmetric about the real axis

• The root locus segments lie on the real axis to the left of an odd number of open loop poles and zeros



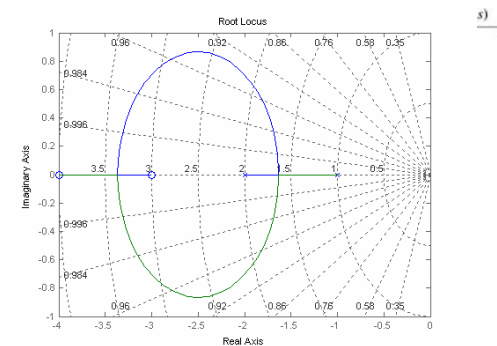
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Rules Continued

• The root locus begins (0 gain) at the poles and ends (∞ gain) at the zeros (finite and infinite) of G(s)H(s)

Recall Ex:

With KG(s)



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Finite and Infinite Poles & Zeros

Consider the open-loop transfer function:

$$\frac{K}{s(s+1)(s+2)}$$

- 3 finite poles at $s = 0$, $s = -1$, and $s = -2$
- No finite zeros, however, consider ...

Every function of s has an equal number of poles and zeros IF we include all finite and infinite poles and zeros!

- But where are these infinite zeros?



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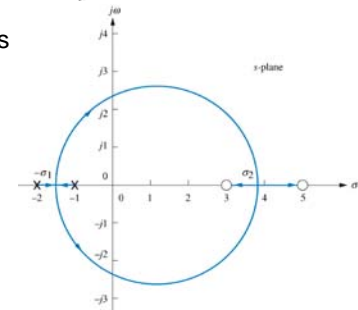
Root Locus Rules Continued

- Asymptotes $\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$
 $\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}$

- Break-out and Break-in points

$$\frac{d[G(s)H(s)]}{ds} = 0$$

$$N(s)\frac{d}{ds}D(s) - \frac{d}{ds}N(s)D(s) = 0$$



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Low Order Loci

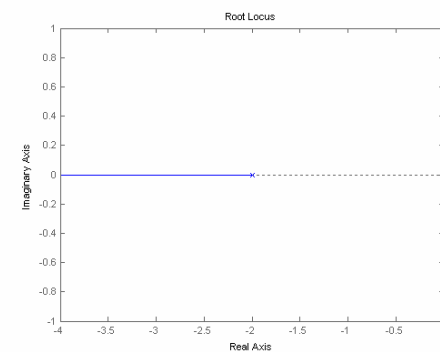
- Use only the first few rules
 - Use the rules in order
- Practice sketching loci to gain proficiency
- The following are some examples of low order loci



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One Pole

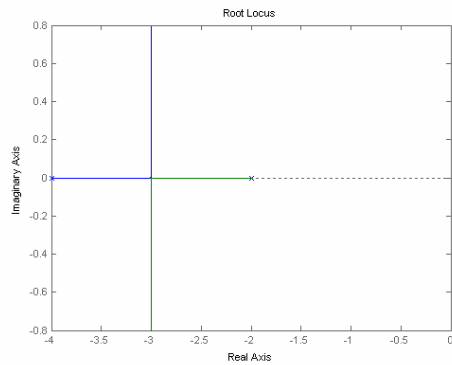
- $1/(s+2)$ w/ pole = -2



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Two Poles

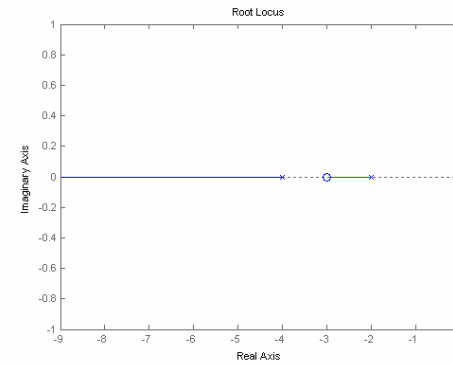
$1/(s^2 + 6s + 8)$ with poles at -2, -4



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One Zero, Two Poles

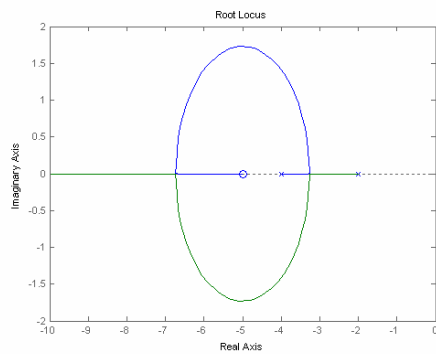
$(s+3)/(s^2 + 6s + 8)$ with one zero at -3 and poles at -2, -4



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Zero Outside Two Poles

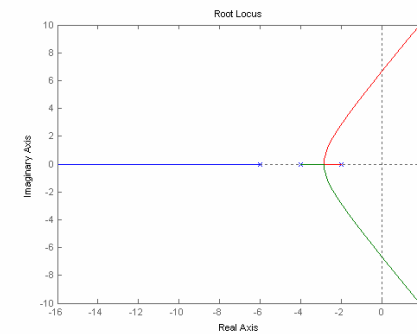
$(s+5)/(s^2 + 6s + 8)$ with one zero at -5 and poles at -2, -4



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Three Poles

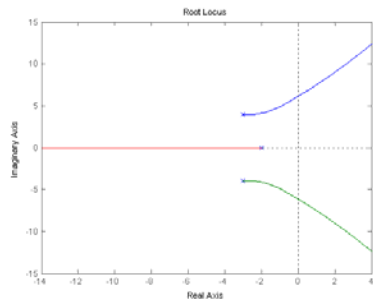
$1/(s^3 + 12s^2 + 44s + 48)$ with poles at -2, -4, -6



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No Breakaway Points

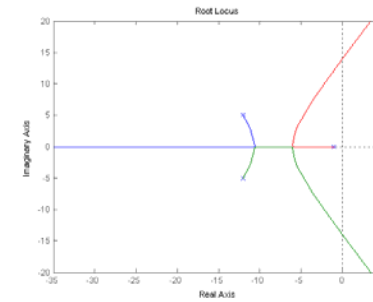
$1/(s^3 + 8s^2 + 37s + 50)$ with poles at $-2, -3 \pm j4$



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With Breakaway Points

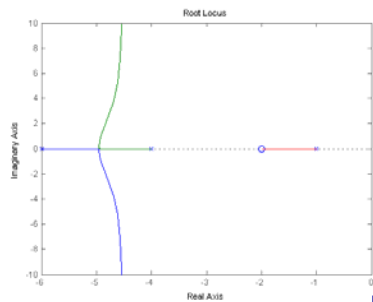
$1/(s^3 + 25s^2 + 193s + 169)$ with poles at $-1, -12 \pm j5$



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One Zero, Three Poles

$(s+2)/(s^3 + 11s^2 + 34s + 24)$ with one zero at -2 and poles at $-1, -4, -6$



Asymptotes

$$\sigma_a = \frac{(-1 - 4 - 6) - (-2)}{3 - 1} = -4.5$$

Breakaway Points

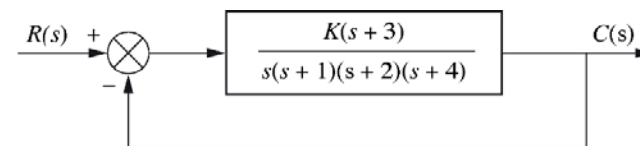
$$\text{root}(2s^3 + 17s^2 + 44s + 44, s) = -4.957$$



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Example 3

Sketch the root locus for the following system



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Example 4

Sketch the root locus for the following system

$$\frac{K}{(s + 2)(s + 4)(s + 6)}$$

