E58: Contro Theory

Lecture Notes for 02/07/2008

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Given the following system,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned} \tag{1}$$

with $\mathbf{x}(0) = \mathbf{x}_0$, we can obtain the closed form solution of $\mathbf{x}(t)$ as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ (\dot{\mathbf{x}} - \mathbf{A}\mathbf{x}) &= \mathbf{B}\mathbf{u} \end{aligned}$$

Multiplying both sides of the equation by $e^{-\mathbf{A}t}$ we get:

$$\begin{aligned} e^{-\mathbf{A}t}(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x}) &= e^{-\mathbf{A}t}\mathbf{B}\mathbf{u} \\ \frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right) &= e^{-\mathbf{A}t}\mathbf{B}\mathbf{u} \\ \int_0^t \frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right) dt &= \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \\ e^{-\mathbf{A}t}\mathbf{x}|_0^t &= \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \\ e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}_0 &= \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \\ e^{-\mathbf{A}t}\mathbf{x}(t) &= \mathbf{x}_0 + \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \end{aligned}$$

Multiplying both sides by $e^{\mathbf{A}t}$, we get

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \\ \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \end{aligned} \tag{2}$$

We call $e^{\mathbf{A}t}$ the *state transition matrix* and the first term of (2) gives the natural response of the system. The second term of (2) is called the *convolution integral* and results in the forced response of the system. Thus, the output of the system is given by

$$\mathbf{y}(t) = \mathbf{C}\left(e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\right) + \mathbf{D}\mathbf{u} \tag{3}$$