

Honors Linear Algebra Placement Problems

Instructions. These problems are for math students who have studied theoretical linear algebra before entering Swarthmore and want to know if they can be placed out of our Honors course Math 28. (Most entering students interested in advanced mathematics take one of 28/28S/28P even if they have had a regular linear algebra course before.)

You may not have any notes or textbooks open while working on these problems. You are welcome to use mathematical software on computers or calculators, but always indicate where and how you have used them.

Put your name and “Honors Linear Algebra Placement” on each page of your solutions. You need not return this question sheet. Please mail your work to the Department of Mathematics and Statistics, 500 College Ave, Swarthmore PA 19081-1390, by August 15.

If you have not placed out of a full year of calculus based on standardized tests, please submit your answers to the problems below *along with* Swarthmore’s Calculus Placement Exam, not instead.

1. For any subsets U, W of a vector space V , define

$$U + W = \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U, \mathbf{w} \in W\}.$$

- a) Prove: if U and W are subspaces of V , then $U + W$ is also subspace of V .
b) Let U be the vector subspace of \mathbb{R}^4 with basis $(1, 1, 1, 2)$ and $(1, -1, 1, -1)$. Let W be the set of vectors (x, y, z, w) that satisfy the system

$$\begin{aligned}x + y - z - w &= 0 \\x - y - z + w &= 0.\end{aligned}$$

- i. Show that $U + W$ is a vector space.
ii. Find a basis for $U + W$.

2. Prove: if T is a linear transformation and also an invertible function, then the inverse function T^{-1} is a linear transformation.

3. In vector terms, the Pythagorean Theorem says that if vectors \mathbf{u} and \mathbf{v} are perpendicular, then

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2.$$

Use dot products to prove that the Pythagorean theorem is true in \mathbb{R}^n , not just \mathbb{R}^2 .

4. Let P be the plane spanned by $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. For each $\mathbf{x} \in \mathbb{R}^4$, let $D(\mathbf{x})$ be the closest point to \mathbf{x} on P . Viewed as a function from \mathbb{R}^4 to \mathbb{R}^4 , D is a linear transformation (you don’t have to prove this). Find its matrix (relative to the standard basis).
5. Let $B = C^{-1}AC$ for some matrix C . Let \mathbf{v} be an eigenvector of A with eigenvalue λ . Show that $C^{-1}\mathbf{v}$ is an eigenvector of B with eigenvalue λ .
6. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be linearly independent vectors in a vector space V . Suppose $T : V \rightarrow V'$ is a one-to-one (that is, injective) linear transformation. Prove that the set of vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)$ is linearly independent.