LAM | ENGINEERING 12 | APRIL 23, 2007

PHYSICAL SYSTEMS ANALYSIS

EXAM 3

1. A physical system has been analyzed and found to have the following input/output differential equation.

$$\frac{d^3y}{dt^3}+3\frac{d^2y}{dt^2}+4\frac{dy}{dt}+2y=\frac{dx}{dt}$$

You may answer in any order that you find convenient; I have listed the questions alphabetically. Each part is worth 8 points.

a. Complete response with input $x(t)=A\cos(\omega t)$, where $\omega = 1$ (assume the initial energy storage conditions are $\ddot{y}(0)=\ddot{y}_0$, with $\ddot{y}(0)=\dot{y}(0)=y(0)=0$; find the form of the response, and then write enough equations to find the unknown constants in your expression, but do ***not*** solve them)

b. Initial value of y if the input is a unit impulse

c. Final value of y if the input is a unit step

d. Particular solution for a sinusoidal input $x(t)=A\cos(\omega t)$, where $\omega=1$

e. Sinusoidal steady state solution

f. Transfer function

g. Transient response (assume the initial energy storage conditions are $\ddot{y}(0)=\ddot{y}_0$, with $\ddot{y}(0)=\dot{y}(0)=y(0)=0$; find the form of the response, and then write enough equations to find the unknown constants in your expression, but do ***not*** solve them)

h. Unit impulse response (assume zero initial conditions before the step is applied)

i. Unit step response (assume zero initial conditions before the step is applied)

j. Zero input response (assume the initial energy storage conditions are $\ddot{y}(0)=\ddot{y}_0$, with $\ddot{y}(0)=\dot{y}(0)=y(0)=0$)

k. Zero state response (assume a unit step input)

EXAM 3

2. (Please note that parts a and b are unrelated. Each part is worth 6 points.)

a. Find y(t) in terms of arbitrary constant coefficients if $Y(s) = \dfrac{s^4 + 2s^3 + 2s^2}{s^2 + 2s + 5}$.

b. If the unit step response of a system is $y(t) = (0.2 + 0.2236\exp(-t)\cos(2t - 2.6779))U(t)$, find the corresponding transfer function for that system.

ENGINEERING 12

PHYSICAL SYSTEMS ANALYSIS

EXAM 3 SOLUTIONS

1. Given $\dddot{y} + 3\ddot{y} + 4\dot{y} + 2y = \dot{x}(t)$ as the I/O diff eq, find

a. Transfer Function $(s^3+3s^2+4s+2)Y(s) = sX(s)$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s^3+3s^2+4s+2}}$$

b. Initial Value $y(0)$ if $x(t) = \delta(t)$ $\quad y(0) = \lim_{s\to\infty} sY(s)$; $Y(s) = H(s)X(s)$

$X(s) = 1$ $\quad y(0) = \lim_{s\to\infty} \frac{s^2}{s^3+3s^2+4s+2} = \boxed{0 = y(0)}$

c. Final Value $y(t\to\infty)$ if $x(t) = U(t)$ $\quad y(t\to\infty) = \lim_{s\to 0} sY(s)$

$X(s) = \frac{1}{s}$ $\quad y(t\to\infty) = \lim_{s\to 0} \frac{s}{s^3+3s^2+4s+2} = \boxed{0 = y(t\to\infty)}$

h. Impulse Response $h(t) = \mathcal{L}^{-1}[H(s)]$ $\quad H(s) = \frac{s}{(s+1)(s^2+2s+2)}$

PFE $\frac{s}{(s+1)[(s+1)^2+1]} = \frac{A}{s+1} + \frac{Bs+C}{(s+1)^2+1} \Rightarrow A = -1,\ B = 1,\ C = 2$

$$\boxed{h(t) = e^{-t}[-1 + \cos t + \sin t]U(t)}$$ (using table)

i. Unit Step Response $y_U(t) = \mathcal{L}^{-1}\left[H(s)\left(\frac{1}{s}\right)\right]$ $\quad \frac{1}{s}H(s) = \frac{1}{(s+1)(s^2+2s+2)}$

PFE $\frac{1}{(s+1)[(s+1)^2+1]} = \frac{A}{s+1} + \frac{Bs+C}{[(s+1)^2+1]} \Rightarrow A = 1,\ B = -1 = C$

$$\boxed{y_U(t) = e^{-t}[1 - \cos t]U(t)}$$ (using tables)

j & k. ZIR & ZSR. Transform original diff eq.

$$s^3Y(s) - s^2y(0) - s\dot{y}(0) - \ddot{y}(0) + 3[s^2Y(s) - sy(0) - \dot{y}(0)]$$
$$+ 4[sY(s) - y(0)] + 2Y(s) = sX(s) - x(0)$$

or $[s^3+3s^2+4s+2]Y(s) = \{(s^2+3s+4)y(0) + (s+3)\dot{y}(0) + \ddot{y}(0)\} + sX(s) - x(0)$

&
$$Y(s) = \underbrace{\frac{(s^2+3s+4)y(0) + (s+3)\dot{y}(0) + \ddot{y}(0)}{s^3+3s^2+4s+2}}_{ZIR} + \underbrace{\frac{sX(s)}{s^3+3s^2+4s+2} - \frac{x(0)}{s^3+3s^2+4s+2}}_{ZSR}$$

since $y(0) = \dot{y}(0) = 0$ & $\ddot{y}(0) = \ddot{y}_0$; $X(s) = \frac{1}{s}$

$$Y(s) = \underbrace{\frac{\ddot{y}_0}{s^3+3s^2+4s+2}}_{\longleftarrow ZIR \longrightarrow} + \underbrace{\frac{1}{s^3+3s^2+4s+2} - \frac{x(0)}{s^3+3s^2+4s+2}}_{\longleftarrow ZSR \longrightarrow}$$

PFEs for these were done in previous section (except for multiplicative constant)

g. Transient Response : set sources = 0 & sub in $y = Ae^{st}$

$$(s^3+3s^2+4s+2)Ae^{st} = (s+1)(s^2+2s+2)Ae^{st} = 0 \Rightarrow s = -1, \; -1 \pm j1$$

$$y_h(t) = A_1 e^{-t} + A_2 e^{-t}\cos(t+\phi)$$

$$\left.\begin{aligned} y_h(0) &= 0 = A_1 + A_2\cos\phi \\ \dot{y}_h(0) &= \dot{0} = -A_1 - A_2\cos\phi - A_2\sin\phi \\ \ddot{y}_h(0) &= \ddot{y}_0 = A_1 + 2A_2\sin\phi \end{aligned}\right\} \text{ 3 eqs ; 3 unknowns}$$

d. & e. Particular Soln for Sinusoidal Input = SSS   Let $\frac{d}{dt} \to j\omega$ in diff eq.

$$(j\omega)^3\underline{Y} + 3(j\omega)^2\underline{Y} + 4(j\omega)\underline{Y} + 2\underline{Y} = j\omega\underline{X} \; ; \quad x(t) = A\cos\omega t \Rightarrow \underline{X} = A\angle 0$$

$$\underline{Y} = \frac{j\omega}{-j\omega^3 - 3\omega^2 + j4\omega + 2} A\angle 0$$

$$\underline{Y} = \frac{A\omega\angle\pi/2}{(2-3\omega^2)+j(4\omega-\omega^3)} = \frac{A\omega}{\sqrt{(2-3\omega^2)^2+(4\omega-\omega^3)^2}} \angle \frac{\pi}{2} - \tan^{-1}\frac{4\omega-\omega^3}{2-3\omega^2} = M\angle\theta$$

$$y(t) = M\cos(\omega t + \theta)$$

& if $\omega = 1$, $\quad y(t) = \frac{A}{\sqrt{10}}\cos(t - 0.3218)$ (angle in rad)

a. Complete Soln $(\omega = 1)$ $\quad y_{comp} = y_h + y_p = y_{ZIR} + y_{ZSR}$ (we have y_h & y_p for sinusoid)

for $t \geq 0$: $y_{comp} = y_h + y_p = A_1 e^{-t} + A_2 e^{-t}\cos(t+\phi) + \frac{A}{\sqrt{10}}\cos(t - 0.3218)$

Apply ICs as in g for 3 eqs to find $A_1, A_2, \& \phi$.

eg $y_{comp}(0) = A_1 + A_2\cos\phi + \frac{A}{\sqrt{10}}\cos(-0.3218) = 0$; etc for $\dot{y}$ & $\ddot{y}$

2. a. Find $y(t)$ if $Y(s) = \frac{s^4+2s^3+2s^2}{s^2+2s+5} = s^2 + 2 + \frac{-2(s+5)}{s^2+2s+5}$ by long ÷

Using table, $y(t) = \frac{d^2}{dt^2}[\delta(t)] + 2\delta(t) + e^{-t}[-2\cos 2t - 6\sin 2t]U(t)$

b. Find $H(s) = \mathcal{L}[h(t)]$ if $y_u(t) = [0.2 + 0.2236e^{-t}\cos(2t - 2.6779)]U(t)$

$$h(t) = \frac{d}{dt}[y_u(t)] = [0.2 + 0.2236\cos(-2.6779)]\delta(t)$$
$$+ U(t)[-0.2236e^{-t}\cos(2t-2.6779) - 0.4472e^{-t}\sin(2t-2.6779)]$$

(first term $\to 0$)

$$H(s) = \mathcal{L}^{-1}[-0.2236e^{-t}\cos(2t-2.6779) - 0.4472e^{-t}\sin(2t-2.6779)]$$

These are in the table! Note $\sin\theta = \cos(\theta - \pi/2)$