Topics in Geometry
Worksheet 9

Unless noted, all problems come from Henle.

1. Analyze each of the following Möbius transformations by finding the fixed points, finding the normal form, and sketching the appropriate coordinate system of Steiner circles indicating the motion of the transformation.
   (a) $Tz = \frac{z}{2z-1}$
   (b) $Tz = \frac{-z}{(1+i)z-i}$.
2. What kind of transformation (elliptic, parabolic, hyperbolic, or loxodromic) is $z \mapsto \frac{1}{z}$? What is its normal form?
3. Find formulas in the form $\frac{az+b}{cz+d}$ for Móbius transformations satisfying the following descriptions:
   (a) an elliptic transformation with fixed points at i and $-i$ and $T(\infty) = 0$.
   (b) a parabolic transformation with fixed point $1 + i$ and $T(\infty) = 100$.
4. A transformation such that $T^2 = I$ is called an involution. Prove that an involutory Möbius transformation must be elliptic. What must λ be for such a transformation?
5. Let the Möbius transformation T take z_1 to w_1 and z_2 to w_2. Prove that T transforms any Steiner circle of the first kind with respect to z_1 and z_2 into a Steiner circle of the first kind with respect to w_1 and w_2.
6. Let the Möbius transformation T take z_1 to w_1 and z_2 to w_2. Prove that T transforms any Steiner circle of the second kind with respect to z_1 and z_2 into a Steiner circle of the second kind with respect to w_1 and w_2.
7. Let C be a cline (generalized circle). Let z and z^* be distinct symmetric points with respect to C. (Recall that Henle uses the word symmetric where Brannan, et al. use the word inverse.) Prove that any cline C' that is orthogonal to C and passes through z must also pass through z^*. Conversely, prove that any cline that passes through z and z^* is orthogonal to C.
8. Let C_1 and C_2 be two nonintersecting clines. Prove that there is a unique pair of points that are simultaneously symmetric to both C_1 and C_2.

9. Verify that the transformations in **H** can be written in the alternative form

$$Tz = \frac{az + b}{\bar{b}z + \bar{a}} \tag{1}$$

where $|a|^2 - |b|^2 = 1$. Conversely, show that such a Möbius transformation is always in **H**.