Topics in Geometry
Worksheet 8

1. Explain the statement "$SO(n)$ is to $O(n)$ as the group of Möbius transformations is to the group of inversive transformations".

2. Show (in detail) that for a Möbius transformation $T(z) = \frac{az+b}{cz+d}$, if $ad - bc$ were allowed to equal zero, the transformation would be constant.

3. Let t be the inversive transformation defined by

$$t(z) = \frac{z+i}{z-i}.$$

   Determine the image of each of the following generalized circles under t:

   (a) the extended line $l \cup \{\infty\}$, where l is the line with equation $y = x$;
   (b) the unit circle.

4. Define a new geometry called Translational Geometry to be the set of properties of the complex plane which are invariant under translations $t(z) = z + c$ where $c \in \mathbb{C}$. Notice that the set of translations is a group so this is a legitimate, if simple, geometry.

   (a) Based on the Fundamental Theorems of Affine, Projective, and Inversive geometries, state the Fundamental Theorem of Translational Geometry.
   (b) Prove your theorem. (Note the similarities between the proofs of all of these theorems.)

5. Find a Möbius transformation:

   (a) sending 1 to 4, 0 to i, and ∞ to -1.
   (b) sending 0 to 0, i to 1, and $-i$ to 2.
   (c) sending 1 to 2, 2 to 3, and 3 to -1.

6. Find a Möbius transformation that takes the circle $|z| = 1$ to the straight line $x + y = 1$.

7. What Möbius transformation represents a rotation of π about the point 1?

8. Using the definition of a Möbius transformation of $\hat{\mathbb{C}}$

$$T(z) = \frac{az+b}{cz+d}$$

   to show that if T is not the identity transformation, then T has exactly one or two fixed points. (Hint: you will need to convince yourself that the quadratic formula works for quadratic equations with complex coefficients.)

9. Prove that a Möbius transformation with a single fixed point at ∞ is a translation.

10. For $a, b, c, d, z \in \mathbb{C}$ prove that

$$\overline{\left(\frac{a\bar{z}+b}{c\bar{z}+d}\right)} = \frac{\bar{a}z+\bar{b}}{\bar{c}z+\bar{d}}.$$