Problem 1) Short answer

a) Convert the following to dB.

\[ \frac{1}{2} \text{ (to the nearest integer)} \quad - 6 \]

\[ \frac{1}{\sqrt{2}} \text{ (to the nearest integer)} \quad - 3 \]

Convert the following from dB:

\[ 6 \text{ dB (to the nearest integer)} \quad 2 \]

\[ 56.4695 \text{ dB (to the nearest integer)} \quad 66 \]

b) We said that at high frequencies the Bode plot of a real pole drops at 20 dB/decade. Some references state the slope as X dB/octave (where an octave is the doubling of the frequency). What is X (to the nearest integer)?

\[ 20 \log_{10}(\omega) \]

\[ 20 \log_{10}(2\omega) = \text{ difference of 6 dB (see above)} \]

\[ \text{Slope} = -6 \text{ dB/octave}. \]
c) Why are Bode plots useful (besides being fun)? Use only one or two brief sentences.

\[ \text{Give you s.s.s. w/o much effort} \]

d) Write the dual of the following statement: "If a voltage source, a resistance, \( R \), and a capacitance, \( C \), are all in series and we take the output to be the voltage across the capacitor, then the circuit exhibits a low-pass characteristic with cutoff frequency of \( 1/(RC) \)." through inductor \( L \)

e) Exhibit that the original statement from part d, as well as your dual statement, are both true. Include schematics of both circuits.

\[ \frac{V_o}{V_i} = \frac{1}{\frac{1}{RC} + 1} = \frac{1}{1 + sRC} \]

\[ \frac{i_o}{i_i} = \frac{1}{\frac{1}{C} + \frac{1}{sC}} = \frac{1}{1 + sLC} \]
Problem 2)  This doesn’t bode well.
Consider the Bode plot below.

![Bode Diagram]

a) Determine if there are poles and zeros at the origin, or any real poles or zeros not at the origin. Determine the location (approximately) of any real poles or zeros.

pole at origin \((-20 \text{ dB dec as } \omega \to 0)\)

real zero at \(-1\) \((\omega_0 = 1 \text{ rad/sec})\)
b) Determine if there are any complex conjugate poles and zeros. Determine (approximately) the values of \( \omega_n \) and \( \zeta \) for any complex conjugate poles and zeros.

\[ \omega_n \approx 3.1 \]

\[ \text{peak height} = 40 \text{dB} = 100 \Rightarrow \frac{1}{2\zeta} \Rightarrow \zeta = 0.005 \]

c) What is the transfer function? Don't forget the constant multiplier (if necessary).

At \( \omega = 1 \) all poles + zeros (in this problem) contribute \( 0 \) dB, so \( k = 20 \text{dB} = 17.5 \)

\[ H(s) = 12.5 \frac{(s^2/1)}{s[(s^2/0.005) + 1]} \]

A
d) If you want the output of the circuit to have the same amplitude as the input, what frequency signal should you use? Use the Bode plot to answer this question. If the input has a magnitude of 2, sketch the input and output below for two cycles (to show phase).

\[
\text{Magnitude} \approx 0 \ \text{dB} \ \text{at} \ \approx 100 \ \text{rad/\s}, \ \phi \approx 180^\circ
\]

![Bode plot diagram]

\[
\frac{10^5}{12 \pm 70^\circ \left( \frac{12^2}{3.1} \right) \pm 180^\circ} = 0.823 \pm 2190^\circ
\]

\[
= -1.57 \pm 180^\circ \approx 180^\circ
\]

Pretty close

---

![Peanuts comic strip]

E11 Final Exam, 2006
Problem 3) \int \Rightarrow s^{-1}!

Consider the circuit shown.

a) Show that the transfer function of the circuit is
\[ \frac{V_o}{V_i} = -\frac{\omega_0}{s} \]
and give \( \omega_0 \) in terms of \( R \) and \( C \).

\[ \frac{V_o}{V_i} = -\frac{\frac{1}{Z_L}}{\frac{1}{sC}} = -\frac{1}{sCR} = -\frac{\omega_0}{s} = H(s) \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

b) Draw a pole-zero diagram of the transfer function.

\[ \star \]

c) Use the results of parts a and b to determine the gain of the circuit as \( \omega \rightarrow 0 \). In other words, what is the output of the circuit due to a unit step input as \( t \rightarrow \infty \).

As \( \omega \rightarrow 0 \), \( \lim_{\omega \rightarrow 0} \Rightarrow \infty \Rightarrow \) the output will be infinite.
d) Use the constitutive relationships for a resistor and capacitor to write an expression for the output voltage as a function of the input voltage. Your result should be in the time domain (i.e., output relative to input, where input = \( V_{in}(t) \)). Hint: sum the currents at the inverting terminal.

\[
\begin{align*}
\frac{v_o}{R} &= \frac{dv_o}{dt} \\
1v_o &= -\frac{1}{\rho_c}v_i + 1t \\
v_o &= -\frac{1}{\rho_c}\int v_i \, dt
\end{align*}
\]

<table>
<thead>
<tr>
<th>Constitutive Relationships</th>
<th>Capacitor</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_R = I_R \cdot R )</td>
<td>( V_C = \frac{1}{C} \int I_C , dt )</td>
</tr>
<tr>
<td>( I_R = \frac{V_R}{R} )</td>
<td>( I_C = C \frac{dV_C}{dt} )</td>
</tr>
</tbody>
</table>

e) Use the result of part d to predict the output as \( t \to \infty \) if the input is a unit step. This should agree with part c.

\[
\begin{align*}
\text{if } v_i & \text{ is unit step} \\
1v_o &= -\frac{1}{\rho_c}t \\
as \quad t \to \infty \quad |v_o| \to \infty
\end{align*}
\]
Problem 4) Not as bad as it looks (though it looks pretty bad).

a) For the circuit shown to the right, all voltages are measured relative to ground. Show that:

\[ V_{\text{out}} = -\left( \frac{Z_1}{Z_1} V_1 + \frac{Z_2}{Z_2} V_2 + \frac{Z_3}{Z_3} V_3 \right) \]

Hint: sum currents at inverting terminal.

\[ -\frac{V_{\text{out}}}{Z_1} = \frac{V_1}{Z_1} + \frac{V_2}{Z_2} + \frac{V_3}{Z_3} \]

Consider the circuit shown below. Don’t let it intimidate you, we’ll take it a little at a time.

b) You should be able to do these with little or no extra computation. Find

i. \( V_a \) in terms of \( V_1, V_2 \) and \( V_3 \). (Hint: use result from part a)

\[ V_a = \left( \frac{1}{sC R_1} V_3 + \frac{1}{sC R_2} V_2 + \frac{1}{sC R_3} V_1 \right) \]

ii. \( V_b \) in terms of \( V_a \). (Hint: you can use result of problem 3, part a (or the results of part a of this problem))

\[ V_b = \frac{1}{sC R} V_a \]

iii. \( V_c \) in terms of \( V_b \). (Hint: this is just the inverting configuration of the op-amp).

\[ V_c = -V_b \]

iv. \( V_c \) in terms of \( V_a \). (Hint: use parts ii and iii of this problem).

\[ V_c = \frac{1}{sC R} V_a \]
We can modify the circuit to make it look scarier (but we know it isn't)

c) Use results from part b to find the transfer function \( \frac{V_a}{V_1} \). Show it is a band-pass transfer function by evaluating it as \( s \to 0 \) and \( s \to \infty \).
(This is the hardest part of this problem, but you can complete the other parts of the problem without doing this one)

\[
V_a = - \left( \frac{1}{sC_R_3} V_a + \frac{V_1}{sC_R_1} + \frac{V_C}{sC_R_2} \right)
\]

\[
\frac{V_C}{sC_R_2} = - \frac{V_b}{sC_R_2} = - \frac{V_C}{s^2 C^2 R_2 R_2}
\]

\[
V_a = - \left( \frac{1}{sC_R_3} V_a + \frac{V_1}{sC_R_1} + \frac{V_a}{s^2 C^2 R_2} \right)
\]

\[
V_a \left( 1 + \frac{1}{sC_R_3} + \frac{1}{s^2 C^2 R_2} \right) = \frac{V_1}{sC_R_1}
\]

\[
\frac{V_a}{V_1} = \frac{s}{s^2 + \frac{s}{C R_3} + \frac{1}{C^2 R_2 R_2}}
\]

\[
\begin{align*}
 s \to 0 & \quad \frac{V_a}{V_1} = 0 \\
 s \to \infty & \quad \frac{V_a}{V_1} = 0
\end{align*}
\]
d) Use results from parts b and c to find the transfer function \( V_c / V_1 \). Show it is a low-pass transfer function by evaluating it as \( s \to 0 \) and \( s \to \infty \).

(If you didn’t get part c, you may assume \( \frac{V_4}{V_1} = H_{\text{imp}} \frac{\frac{\omega_0}{Q}}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \) to complete this part)

\[
\frac{V_c}{V_1} = \frac{V_v}{V_i} \cdot \frac{V_c}{V_a} = \frac{V_v}{V_i} \cdot \frac{1}{s \alpha} = \frac{1}{s^2 + \frac{s}{C R} + \frac{1}{C^2 R^2}}
\]

\( s \to 0 \) \quad \frac{V_c}{V_1} \to \frac{R_c}{R_1}

\( s \to \infty \) \quad \frac{V_c}{V_1} \to 0

\( e) \) For chosen values of circuit components we get:

\[
\frac{V_c}{V_1} = \frac{1.1 \cdot 10^6}{s^2 + 110s + 1000}
\]

Determine the constituent parts of the Bode plot (i.e., locations of poles and zeros, multiplying constant...). Sketch Bode Plot on next page (I included 2 sets of axes in case you make a mistake).

\[
\begin{align*}
&= -\frac{1.1 \times 10^6}{(s + 100)(s + 1000)} = \frac{\frac{1 \times 10^6}{1000}}{(s + \frac{5}{10})(1 + \frac{5}{1000})} \\
&= -1.1 \times 10^3 \quad \frac{1}{(1 + \frac{s}{10})(1 + \frac{s}{1000})}
\end{align*}
\]

\( 1.1 \times 10^3 \to 61 \, \text{dB} \)

See page 13
**Problem 5**  Consider the circuit shown below. Find the current $i(t)$ (through $R_1$) for all time if there is no initial energy storage in the circuit, and the voltage source is an impulse $v_s(t) = V_0 \delta(t)$ V. (Note that the last V in the previous expression represents the units, volts).

$$i(t)$$

![Circuit Diagram]

**UNIT STEP**

$$T = (R_1 R_2) C$$

$$v(0^+) = 0$$

$$u(\infty) = \frac{R}{R_1 + R_2}$$

**UNIT STEP RESPONSE**

$$v_0(t) = \left( \frac{R}{R_1 + R_2} + \frac{R}{R + R_2} \right) e^{-t/T} V_0 + R\left( 1 - e^{-t/T} \right) u(t)$$

**UNIT IMPULSE RESPONSE**

$$v_b(t) = \frac{R}{R_1 R_2} \left[ \frac{1}{T} e^{-t/T} u(t) + (1 - e^{-t/T}) \delta(t) \right]$$

$$v_c(t) = \frac{1}{R_1} e^{-t/T} u(t) = \frac{1}{R_1} e^{-t/T} V_0$$

if $v_c = V_0 \delta(t)$

$$v_{out} = \frac{V_0}{R_2} e^{-t/T} u(t)$$

**Solved... Problem asks for $i(t)$**

$$i(t) = \frac{v(t) - v_0}{R} = V_0 \delta(t) - \frac{V_0}{RC} e^{-t/T} u(t)$$

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E11 Final Exam, 2006
Problem 6) Consider the circuit shown below. It has been connected for a long time. If \( i_1(t) = 0.7454 \cos(100t + 1.249) \text{ A} \), find \( v(t) \).

\[
\begin{align*}
\sigma - 3i_1 - 3j(i_1 - i_2) &= 0 \\
v &= i_1(3 + 3j) - 3j i_2 \\
\sigma - 3i_1 - 3j(i_1 - i_2) &= 0 \\
v &= 0.7454 \cos(100t + 1.249) \\
\end{align*}
\]

\[
\begin{align*}
3j(i_1 - i_2) &= 3j i_1 + 2j i_2 \\
\sigma - 3i_1 + 3j i_2 &= 0 \\
\end{align*}
\]

\[
\begin{align*}
i_1(3 + 3j) &= i_2 j \\
(-3 + 3j) &= i_1(3 + 3j) \\
\end{align*}
\]

\[
\begin{align*}
i_2 &= i_1 \left( \frac{-3 + 3j}{j} \right) \\
i_1(3 + 3j) &= i_1(3 + 3j) \\
\end{align*}
\]

\[
\begin{align*}
v &= i_1(3 + 3j) - 3j(i_1(3 + 3j)) \\
&= i_1(3 + 3j) - 9j i_1 + 9i_1 \\
&= i_1(12 - 6j) = i_1(13.416 \angle -46.36^\circ) \\
&= (0.7454 \angle 1.249) (13.416 \angle -46.36^\circ) \\
&= 10 \angle 78.54^\circ = 10 \angle 45^\circ \\
\end{align*}
\]

\[
\begin{align*}
v(t) &= 10 \cos(100t + 78.54) = 10 \cos(100t + 45^\circ) \\
\end{align*}
\]
## Rules for Making Bode Plots

<table>
<thead>
<tr>
<th>Term</th>
<th>Magnitude</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Constant: K</strong></td>
<td>$20 \cdot \log_{10}(</td>
<td>K</td>
</tr>
<tr>
<td><strong>Real Pole:</strong> $\frac{1}{s + \frac{1}{\omega_0}}$</td>
<td>- Low freq. asymptote at 0 dB</td>
<td>$K &lt; 0$: ±180°</td>
</tr>
<tr>
<td></td>
<td>- High freq. asymptote at -20 dB/dec</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Connect lines at $\omega_0$</td>
<td></td>
</tr>
<tr>
<td><strong>Real Zero:</strong> $\frac{s}{\omega_0} + 1$</td>
<td>- Low freq. asymptote at 0 dB</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- High freq. asymptote at +20 dB/dec.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Connect lines at $\omega_0$</td>
<td></td>
</tr>
<tr>
<td><strong>Pole at Origin:</strong> $\frac{1}{s}$</td>
<td>-20 dB/dec; through 0 dB at $\omega=1$</td>
<td>-90°</td>
</tr>
<tr>
<td><strong>Zero at Origin:</strong> $s$</td>
<td>+20 dB/dec; through 0 dB at $\omega=1$</td>
<td>+90°</td>
</tr>
<tr>
<td><strong>Underdamped Poles:</strong></td>
<td>- Low freq. asymptote at 0 dB</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- High freq. asymptote at -40 dB/dec.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Draw peak† at freq. $\omega_c = \frac{\omega_0}{\sqrt{1 - 2\zeta^2}}$ with amplitude</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$H(j\omega_c) = -20 \cdot \log_{10}(2\zeta\sqrt{1 - \zeta^2})$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Connect lines</td>
<td></td>
</tr>
<tr>
<td><strong>Underdamped Zeros:</strong></td>
<td>- Draw low freq. asymptote at 0 dB</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Draw high freq. asymptote at +40 dB/dec.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Draw dip‡ at freq. $\omega_c = \frac{\omega_0}{\sqrt{1 - 2\zeta^2}}$ with amplitude</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$H(j\omega_c) = +20 \cdot \log_{10}(2\zeta\sqrt{1 - \zeta^2})$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- Connect lines</td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**

* Rules for drawing zeros create the mirror image (around 0 dB, or 0°) of those for a pole with the same $\omega_0$.
† For underdamped poles and zeros peak exists only for $0 < \zeta < 0.707 = \frac{1}{\sqrt{2}}$ and peak freq. is typically very near $\omega_0$.
‡ For underdamped poles and zeros if $\zeta < 0.02$ draw phase vertically from 0 to -180 degrees at $\omega_0$.

For $n^{th}$ order pole or zero make asymptotes, peaks and slopes in times higher than shown (i.e., second order asymptote is -40 dB/dec, and phase goes from 0 to -180°). Don’t change frequencies, only plot values and slopes.
Quick Reference for Making Bode Plots

If starting with a transfer function of the form (some of the coefficients $b_i$, $a_i$ may be zero).

$$H(s) = C \frac{s^n + \cdots + b_i s + b_0}{s^m + \cdots + a_i s + a_0}$$

Factor polynomial into real factors and complex conjugate pairs ($p$ can be positive, negative, or zero; $p$ is zero if $a_0$ and $b_0$ are both non-zero).

$$H(s) = C \cdot s^p \frac{(s + \omega_{z1})(s + \omega_{z2}) \cdots (s^2 + 2\zeta_{z1}\omega_{0z1}s + \omega_{0z1}^2)(s^2 + 2\zeta_{z2}\omega_{0z2}s + \omega_{0z2}^2) \cdots}{(s + \omega_{p1})(s + \omega_{p2}) \cdots (s^2 + 2\zeta_{p1}\omega_{0p1}s + \omega_{0p1}^2)(s^2 + 2\zeta_{p2}\omega_{0p2}s + \omega_{0p2}^2) \cdots}$$

Put polynomial into standard form for Bode Plots.

$$H(s) = C \frac{\omega_{z1} \omega_{z2} \cdots \omega_{0z1} \omega_{0z2} \cdots}{\omega_{p1} \omega_{p2} \cdots \omega_{0p1} \omega_{0p2} \cdots} \cdot s^p \frac{\left(\frac{s}{\omega_{z1}} + 1\right) \left(\frac{s}{\omega_{z2}} + 1\right) \cdots \left(\frac{s}{\omega_{0z1}} + 1\right)}{\left(\frac{s}{\omega_{p1}} + 1\right) \left(\frac{s}{\omega_{p2}} + 1\right) \cdots \left(\frac{s}{\omega_{0p1}} + 1\right)} \left(\frac{s}{\omega_{0z1}} + 1\right) \left(\frac{s}{\omega_{0z2}} + 1\right) \cdots \left(\frac{s}{\omega_{0p1}} + 1\right) \left(\frac{s}{\omega_{0p2}} + 1\right) \cdots$$

$$= K \cdot s^n \frac{\left(\frac{s}{\omega_{z1}} + 1\right) \left(\frac{s}{\omega_{z2}} + 1\right) \cdots \left(\frac{s}{\omega_{0z1}} + 1\right) + 2\zeta_{z1} \left(\frac{s}{\omega_{0z1}} + 1\right) + \frac{s}{\omega_{0z2}} + 1\right) \left(\frac{s}{\omega_{0z2}} + 1\right) \cdots \left(\frac{s}{\omega_{0p1}} + 1\right) \left(\frac{s}{\omega_{0p2}} + 1\right) \cdots$$

Take the terms (constant, real poles and zeros, origin poles and zeros, complex poles and zeros) one by one and plot magnitude and phase according to rules on previous page. Add up resulting plots.
Can I help you?

Yes, this is Pig, and I'd like to complain about your testing procedure.

What about it?

Well, I'd do a lot better on your tests if I had some advance notice... I'm no good with pop quizzes.

... and I don't know if it's a problem on your end or mine, but I can't even hear the stupid questions.

All I hear is this long monotone sound... BEEEEEEEEEEEP... I think it's drowning out the questions.

I'm sure if you cleared up these few things, I'd pass, and you wouldn't have to keep giving me the stupid test.

Sir, you can't "pass" a test of our emergency broadcast system.

I really don't need your pessimism, sir.