E12 Exam 1, 2008

Name: ANSWER (3 pts)

Complete all three problems. All count equally towards your grade. If you have questions, please ask me; I didn't intend for the wording of any of the problems to be unclear. Some parts are much easier than others; if a part of a problem seems easy, it probably is. If you get stuck, move on; many parts can be solved without completing earlier parts.

Good luck!

Problem 1

a) Briefly (one or two sentences) explain why Laplace Transforms are useful.

Differential equations become algebra – easier.
Convolution becomes multiplication – easier
Initial conditions at $t=0^-$

b) Find y(t) if $Y(s)=\frac{s^2+7s+6}{s\left(s^2+5s+6\right)}$ $= \frac{1}{s}+\frac{2}{s+2}-\frac{2}{s+3}$

$$y(t)=\left(1+2e^{-t}-2e^{-3t}\right)u(t)$$

c) Verify your answer with the initial value theorem and the final value theorem.

$y(0)=\lim_{s\to\infty} sY(s)=1$ $\qquad y(0)=1$ ✓

$y(\infty)=\lim_{s\to 0} sY(s)=\frac{6}{6}=1$ $\qquad y(\infty)=1$ ✓

d) Given

$$a_0\ddot{y}(t)+a_1\dot{y}(t)+a_2y(t)=b_0\ddot{x}(t)+b_1\dot{x}(t)+b_2x(t)$$

Find an expression for the zero-state solution, $Y_{ZS}(s)$, in terms of X(s) and the "a" and "b" coefficients.

$$Y_{ZS}(s)=\frac{b_0s^2+b_1s+b_2}{a_0s^2+a_1s+a_2}X(s)$$

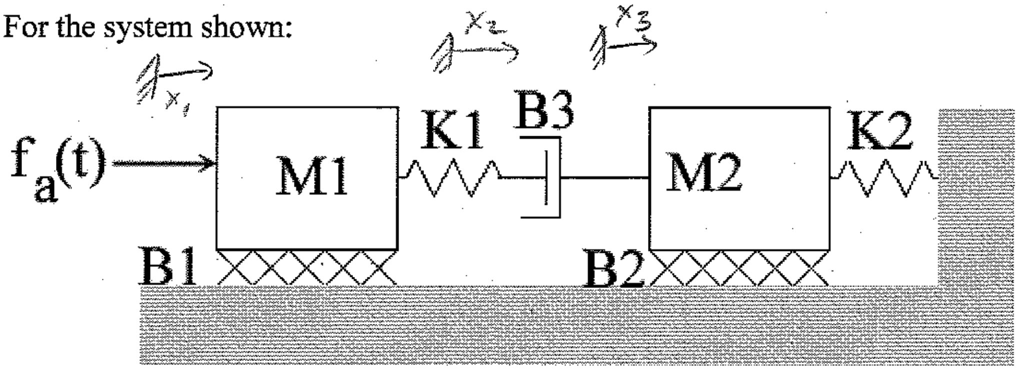
e) For the differential equation in part d, find an expression for the zero-input solution, $Y_{ZI}(s)$, in terms of initial conditions $(\dot{y}(0),\ y(0))$ and the "a" and "b" coefficients.

$$Y_{ZI}(s)=\frac{a_0sy(0)+a_1\dot{y}(0)+a_1y(0)}{a_0s^2+a_1s+a_2}$$

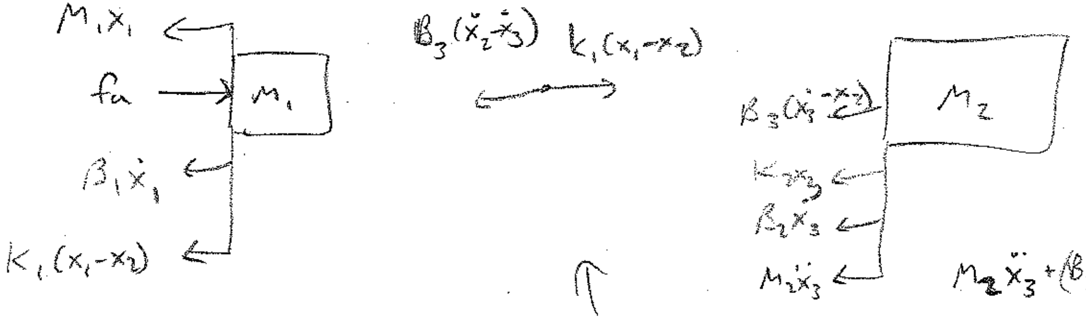
f) For the differential equation in part d, find an expression for the impulse response, H(s), in terms of the "a" and "b" coefficients.

$$H(s)=\frac{b_0s^2+b_1s+b_2}{a_0s^2+a_1s+a_2}$$

Problem 2) For the system shown:



a) Draw appropriate free body diagrams and write the corresponding equations of motion. Make sure your position variables are clearly defined on the picture above.



$M_2\ddot{x}_3+(B_2+B_3)\dot{x}_3+k_2x_3-B_3\dot{x}_2=0$

$M_1\ddot{x}_1+B_1\dot{x}_1+k_1x_1-k_2x_2=f_a$

$B_3\dot{x}_2+k_1x_2-B_3\dot{x}_3-k_1x_1=0$

b) Let $f_a(t)$ be the input and the position of mass M2 be the output. You are told that the impulse response of the system is some function h(t). How would you find the output if the system was starting from rest and the input was a ramp for t>0 ($f_a(t)=tu(t)$) without using Laplace Transforms? Be specific – give as specific an equation as you are able.

$$y(t) = h(t) * f_a(t) = h(t) * t = \int_0^t h(\lambda)(t-\lambda)\,d\lambda$$

$$= \int_0^t h(t-\lambda)\,\lambda\,d\lambda$$

c) In the situation from part b, how would you find the output using Laplace Transforms. Be specific – give as specific an equation as you are able.

$$Y(s) = H(s) \cdot F_a(s) = H(s)\frac{1}{s^2}$$

d) (unrelated to parts a through c). What is y(t) if

$$\left(Y(s)=\frac{s^3+3s^2+4s+20}{s^2\left(s^2+4s+20\right)} = \frac{A}{s^2}+\frac{B}{s}+\frac{Cs+D}{(s+2)^2+16} \right) s^2(s^2+4s+20)$$

$$s^3+3s^2+4s+20 = A(s^2+4s+20)+Bs(s^2+4s+20)+(Cs+D)s^2$$

$$\begin{aligned} s^3 &\Rightarrow & 1 &= B+C & & C=1 \\ s^2 &\Rightarrow & 3 &= A+4B+D & & D=2 \nearrow \\ s^1 &\Rightarrow & 4 &= 4A+20B & & B=0 \nearrow \\ s^0 &\Rightarrow & 20 &= 20A & & A=1 \nearrow \end{aligned}$$

$$Y(s)=\frac{A}{s^2}+\frac{s+2}{(s+2)^2+4^2}$$

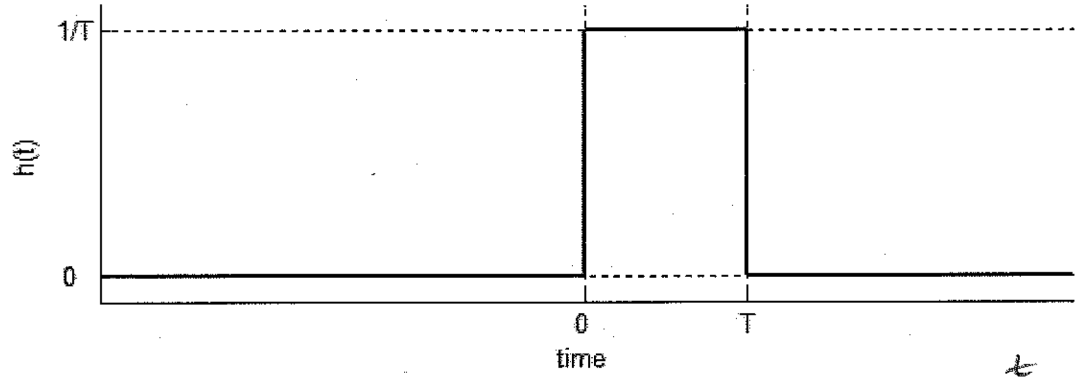
$$y(t)=\left(t+e^{-2t}\cos(4t)\right)u(t)$$

Problem 3) Taking the average – Laplace Style.

In one of your homework problems you showed that given h(t) (below) that the convolution of a function x(t) and h(t), x(t)*h(t), is equal to the average of x(t) over the last T seconds. You did this by looking at the convolution integral. In this problem we will do this using Laplace Transforms. Don't worry, even if you didn't get the homework problem, you can finish this problem. In short: if y(t) is the average of x(t) over the last T seconds, then

y(t)=x(t)*h(t).

The function h(t) is specified ~~in problem 1d.~~ below.



a) Give expressions for H(s) and X(s) if h(t) is as given above and x(t)= cos($\overset{t}{\cancel{2\pi f t}}$)u(t).

$H(s) = \frac{1}{T}\left(\frac{1}{s} - \frac{1}{s}e^{-Ts}\right)$ $x(t) = \frac{s}{s^2+1}$

b) If y(t)=x(t)*h(t), use H(s) and X(s) to find Y(s).

$$Y(s) = H(s)X(s) = \frac{1}{T}\left(\frac{1}{s}\frac{s}{s^2+1} - \frac{1}{s}\frac{s}{s^2+1}e^{-Ts}\right)$$

$$= \frac{1}{T}\left(\frac{1}{s^2+1} - \frac{1}{s^2+1}e^{-Ts}\right)$$

c) Use Y(s) to find an expression for y(t).

$$y(t) = \frac{1}{T}\left(\sin(t) - \sin(t-T)\,u(t-T)\right)$$

d) Find y(t) for t>T. For what values of T is y(t)=0 for t>T. Why?

for $t > t$ $y(t) = \frac{1}{T}\left(\sin(t) - \sin(t-T)\right)$

$= 0$ for $T = 2\pi k$

if $T = 2\pi k$ we are integrating over an integer multiple of wavelengths

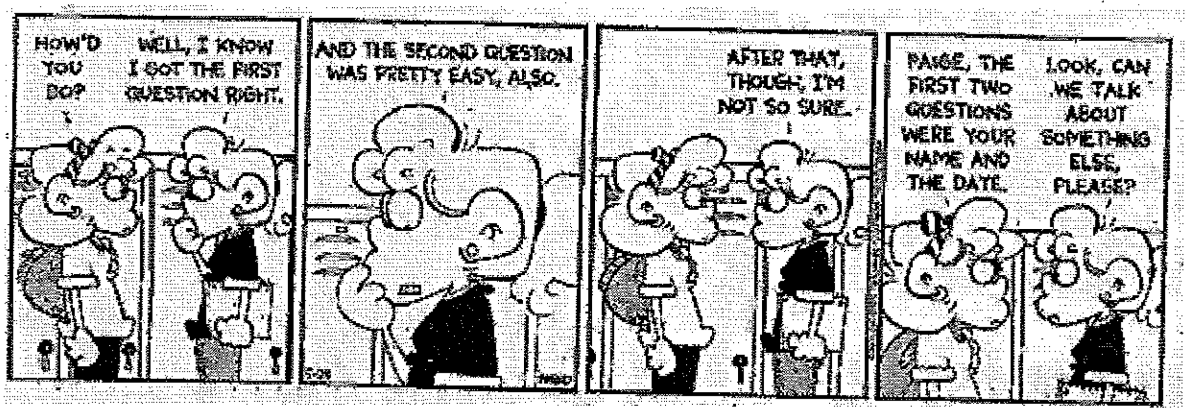
Common Laplace Transform Pairs

Time Domain Function		**Laplace Domain Function**
Name	Definition*	
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$\frac{1}{s}$
Unit Ramp	t	$\frac{1}{s^2}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Asymptotic Exponential	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
Dual Exponential	$\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$	$\frac{1}{(s+a)(s+b)}$
Asymptotic Dual Exponential	$\frac{1}{ab}\left[1+\frac{1}{a-b}\left(be^{-at}-ae^{-bt}\right)\right]$	$\frac{1}{s(s+a)(s+b)}$
Time multiplied Exponential	te^{-at}	$\frac{1}{(s+a)^2}$
Sine	$\sin(\omega_n t)$	$\frac{\omega}{s^2+\omega_n^2}$
Cosine	$\cos(\omega_n t)$	$\frac{s}{s^2+\omega_n^2}$
Decaying Sine	$e^{-at}\sin(\omega_n t)$	$\frac{\omega}{(s+a)^2+\omega_n^2}$
Decaying Cosine	$e^{-at}\cos(\omega_n t)$	$\frac{s+a}{(s+a)^2+\omega_n^2}$
Generic Oscillatory Decay	$e^{-at}\left[B\cos\left(\omega_n t\right)+\frac{C-aB}{\omega_n}\sin\left(\omega_n t\right)\right]$	$\frac{Bs+C}{\left(s+a\right)^2+\omega_n^2}$
Prototype Second Order Lowpass, underdamped	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
Prototype Second Order Lowpass, underdamped - Step Response	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\phi\right)$ $\phi=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$

*All time domain functions are implicitly=0 for t<0 (i.e. they are multiplied by unit step).

Common Laplace Transform Properties

Name	Illustration
Definition of Transform	$\mathrm{f(t)} \xleftrightarrow{\mathrm{L}} \mathrm{F(s)}$ $\mathrm{F(s)} = \int_{0^-}^{\infty} \mathrm{f(t)e^{-st}dt}$
Linearity	$Af_1(t)+Bf_2(t) \xleftrightarrow{L} AF_1(s)+BF_2(s)$
First Derivative	$\frac{df(t)}{dt} \xleftrightarrow{L} sF(s)-f(0^-)$
Second Derivative	$\frac{d^2 f(t)}{dt^2} \xleftrightarrow{L} s^2F(s)-sf(0^-)-\dot{f}(0^-)$
n^{th} Derivative	$\frac{d^n f(t)}{dt^n} \xleftrightarrow{L} s^nF(s)-\sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0^-)$
Integral	$\int_0^t f(\lambda)d\lambda \xleftrightarrow{L} \frac{1}{s}F(s)$
Time Multiplication	$tf(t) \xleftrightarrow{L} -\frac{dF(s)}{ds}$
Time Delay	$\mathrm{f(t-a)U(t-a)} \xleftrightarrow{\mathrm{L}} \mathrm{e^{-as}F(s)}$
Complex Shift	$f(t)e^{-at} \xleftrightarrow{L} F(s+a)$
Scaling	$f\left(\frac{t}{a}\right) \xleftrightarrow{L} aF(as)$
Convolution Property	$f_1(t)*f_2(t) \xleftrightarrow{L} F_1(s)F_2(s)$
Initial Value	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$
Final Value (if final value exists)	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$



```
%% E12 Take home exam
% Use "File->Publish to html" to make the code clear

%% part a
syms s t
H=0.5*s/(s^2+0.5*s+1);
disp('H(s)=');
pretty(H)

%% part b
h=simple(ilaplace(H));
disp('h(t)=');
pretty(vpa(h,3))



%%
%
% $$e^{-at}(Acos(\omega t)+Bsin(\omega t)) = Me^{-at}cos(\omega t+\phi)$$
%
M=sqrt((0.033*15)^2+(0.033*3.87)^2);
phi=atan2(3.87,15);  %phi in degrees.
disp('We can also write as cosine with phase shift')
disp(['h(t)=' num2str(M) '*e^(-0.25*t)*cos(0.968t+' num2str(phi) ') (rad)'])
disp(['h(t)=' num2str(M) '*e^(-0.25*t)*cos(0.968t+' num2str(phi*180/pi) ') (deg)'])



%% part c
[nsym,dsym]=numden(H);
d=sym2poly(dsym);
disp('Poles of H(s)=');
disp(roots(d));

%% part d
F=laplace(cos(t));
Y=F*H;
y=simple(ilaplace(Y));
disp('y(t)=');
pretty(vpa(y,3))
%%
% The first term is the only one that doesn't decay, so for large t,
% y(t)=cos(t).

%% part e
F=laplace(cos(5*t));
Y=F*H;
y=simple(ilaplace(Y));
disp('y(t)=');
pretty(vpa(y,3))
%%
% The first two terms don't decay, so for large t,
% y(t)=0.011cos(5t)+0.103sin(5t).
```

```
%%
%
% $$Acos(\omega t)+Bsin(\omega t) = Mcos(\omega t+\phi)$$
%
M=sqrt(0.011^2+0.103^2);
phi=atan2(-0.103,0.011);  %phi in degrees.
disp('We can also write as cosine with phase shift')
disp(['y(t)=' num2str(M) '*cos(5t+' num2str(phi) ') (rad)'])
disp(['y(t)=' num2str(M) '*cos(5t+' num2str(phi*180/pi) ') (deg)'])
%%
% This y(t) has an amplitude of only 0.104 (as compared to an amplitude of
% 1 for part d).

%% part f
F=laplace(cos(0*t));
Y=F*H;
y=simple(ilaplace(Y));
disp('y(t)=');
pretty(vpa(y,3))
%%
% There is no term that doesn't decay, so the output at long time is zero.
% This is not unexpected, the differential equation show that it
% differentiates the input, since the input is constant (for t>0) it's
% derivative is zero, so there is nothing driving the input for t>0, so
% the output decays to nothing.

%% part g
% Y(s) for part d has two poles at s=+/-j.  At that point the H(s) surface
% has a magnitude equal to 1, so the magnitude of that pole is unchanged -
% the cosine has a
s=j;
Hval=eval(H);
disp(['H(s) evaluated at s=' num2str(s) ' = ' num2str(Hval)]);
disp(['Mag = ' num2str(abs(Hval))]);

%%
% Y(s) for part e has two poles at s=+/-5j.  At that point the H(s) surface
% is quite small, so the pole is multiplied by a small number, and the
% output at that frequency is small.
s=5*j;
Hval=eval(H);
disp(['H(s) evaluated at s=' num2str(s) ' = ' num2str(Hval)]);
disp(['Mag = ' num2str(abs(Hval))]);

%%
% Y(s) for part f has a pole at s=0.  At that point the H(s) surface is
% equal to zero, so the pole is negated, and the output at that frequency
% is zero.
s=0;
Hval=eval(H);
disp(['H(s) evaluated at s=' num2str(s) ' = ' num2str(Hval)]);
disp(['Mag = ' num2str(abs(Hval))]);
```

E12 Take home exam

Use "File->Publish to html" to make the code clear

Contents

part a

```
syms s t
H=0.5*s/(s^2+0.5*s+1);
disp('H(s)=');
pretty(H)
```

```
H(s)=

                                        s
                              1/2 --------------
                                   2
                                  s  + 1/2 s + 1
```

part b

```
h=simple(ilaplace(H));
disp('h(t)=');
pretty(vpa(h,3))
```

```
h(t)=

          0.0333 exp(-0.250 t) (15. cos(0.968 t) - 3.87 sin(0.968 t))
```

$$e^{-at}(Acos(\omega t)+Bsin(\omega t)) = Me^{-at}cos(\omega t+\phi)$$

```
M=sqrt((0.033*15)^2+(0.033*3.87)^2);
phi=atan2(3.87,15);  %phi in degrees.
disp('We can also write as cosine with phase shift')
disp(['h(t)=' num2str(M) '*e^(-0.25*t)*cos(0.968t+' num2str(phi) ') (rad)'])
disp(['h(t)=' num2str(M) '*e^(-0.25*t)*cos(0.968t+' num2str(phi*180/pi) ') (deg)'])
```

```
We can also write as cosine with phase shift
h(t)=0.51121*e^(-0.25*t)*cos(0.968t+0.25249) (rad)
h(t)=0.51121*e^(-0.25*t)*cos(0.968t+14.4668) (deg)
```

part c

```
[nsym,dsym]=numden(H);
d=sym2poly(dsym);
disp('Poles of H(s)=');
disp(roots(d));
```

```
Poles of H(s)=
  -0.2500 + 0.9682i
  -0.2500 - 0.9682i
```

part d

```
F=laplace(cos(t));
Y=F*H;
y=simple(ilaplace(Y));
disp('y(t)=');
pretty(vpa(y,3))
```

```
y(t)=

     cos(t) + 0.0667 exp(-0.250 t) (-15. cos(0.968 t) + 3.87 sin(0.968 t))
```

The first term is the only one that doesn't decay, so for large t, y(t)=cos(t).

part e

```
F=laplace(cos(5*t));
Y=F*H;
y=simple(ilaplace(Y));
disp('y(t)=');
pretty(vpa(y,3))
```

```
y(t)=

  0.0107 cos(5. t) + 0.103 sin(5. t)

         - 0.0000286 exp(-0.250 t) (375. cos(0.968 t) + 646. sin(0.968 t))
```

The first two terms don't decay, so for large t, y(t)=0.011cos(5t)+0.103sin(5t).

$$Acos(\omega t) + Bsin(\omega t) = Mcos(\omega t + \phi)$$

```
M=sqrt(0.011^2+0.103^2);
phi=atan2(-0.103,0.011);  %phi in degrees.
disp('We can also write as cosine with phase shift')
disp(['y(t)=' num2str(M) '*cos(5t+' num2str(phi) ') (rad)'])
disp(['y(t)=' num2str(M) '*cos(5t+' num2str(phi*180/pi) ') (deg)'])
```

```
We can also write as cosine with phase shift
y(t)=0.10359*cos(5t+-1.4644) (rad)
y(t)=0.10359*cos(5t+-83.9041) (deg)
```

This y(t) has an amplitude of only 0.104 (as compared to an amplitude of 1 for part d).

part f

```
F=laplace(cos(0*t));
Y=F*H;
y=simple(ilaplace(Y));
disp('y(t)=');
pretty(vpa(y,3))
```

```
y(t)=

                       0.515 exp(-0.250 t) sin(0.968 t)
```

There is no term that doesn't decay, so the output at long time is zero. This is not unexpected, the differential equation show that it differentiates the input, since the input is constant (for t>0) it's derivative is zero, so there is nothing driving the input for t>0, so the output decays to nothing.

part g

Y(s) for part d has two poles at s=+/-j. At that point the H(s) surface has a magnitude equal to 1, so the magnitude of that pole is unchanged - the cosine has a

```
s=j;
Hval=eval(H);
disp(['H(s) evaluated at s=' num2str(s) ' = ' num2str(Hval)]);
disp(['Mag = ' num2str(abs(Hval))]);
```

```
H(s) evaluated at s=0+1i = 1
Mag = 1
```

Y(s) for part e has two poles at s=+/-5j. At that point the H(s) surface is quite small, so the pole is multiplied by a small number, and the output at that frequency is small.

```
s=5*j;
Hval=eval(H);
disp(['H(s) evaluated at s=' num2str(s) ' = ' num2str(Hval)]);
disp(['Mag = ' num2str(abs(Hval))]);
```

```
H(s) evaluated at s=0+5i = 0.010734-0.10305i
Mag = 0.10361
```

Y(s) for part f has a pole at s=0. At that point the H(s) surface is equal to zero, so the pole is negated, and the output at that frequency is zero.

```
s=0;
Hval=eval(H);
disp(['H(s) evaluated at s=' num2str(s) ' = ' num2str(Hval)]);
disp(['Mag = ' num2str(abs(Hval))]);
```

```
H(s) evaluated at s=0 = 0
Mag = 0
```

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