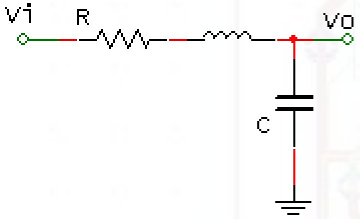
2nd Order Transient Response

- Start with a second order circuit (typically an inductor and a capacitor, or two of each).



- We get a second order differential equation.

$$\frac{d^2 v_o(t)}{dt^2} + \frac{R}{L}\frac{dv_o(t)}{dt} + \frac{1}{LC}v_o(t) = \frac{1}{LC}v_i(t)$$

- First order method from last lab is not helpful, but methods from class are!
- Assume homogeneous response is of the form Ae^{st}.
- Find characteristic values (“s”); for second order circuit we expect two.
- For constant input (this lab) assume particular solution is a constant.

Homogeneous Response

- Rewrite differential equation (with no input):

$$\frac{d^2v_o(t)}{dt^2}+\frac{R}{L}\frac{dv_o(t)}{dt}+\frac{1}{LC}v_o(t)=0$$

$$\frac{d^2v_o(t)}{dt^2}+2\zeta\omega_0\frac{dv_o(t)}{dt}+\omega_0^2v_o(t)=0$$

- Assume homogeneous response is of the form Ae^{st}.

$$s^2+2\zeta\omega_0s+\omega_0^2=0$$

$$s_{1,2}=\frac{-2\zeta\omega_0\pm\sqrt{(2\zeta\omega_0)^2-4\omega_0^2}}{2}$$

$$=-\zeta\omega_0\pm\omega_0\sqrt{\zeta^2-1}$$

- Three cases of interest with passive circuits.

$\zeta>1:$ two distinct real roots, overdamped

$\zeta=1:$ two distinct real roots, critically damped

$0\leq\zeta<1:$ two distinct complex roots, underdamped

- We will consider only first (overdamped) and third (underdamped) cases.

Overdamped Case (ζ>1)

This is the easy case (for calculations, but not in lab)

- Find roots.

$$s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

- Write homogeneous response (constants are unknown).

$$v_{o,h}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$= A_1 e^{t/\tau_1} + A_2 e^{t/\tau_2}$$

$$\tau_{1,2} = \frac{1}{-\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}}$$

Underdamped Case (0≤ζ<1)

This is the easy case (in lab, but not for calculations)

- Find roots.

$$\begin{aligned} s_{1,2} &= -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1} \\ &= -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2} \\ &= -\alpha \pm j\omega_d \qquad \text{(book's notation)} \end{aligned}$$

- Write homogeneous response. There are several forms, use whichever is convenient (constants are unknown).

$$v_{o,h}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad (A_1, A_2 \text{ complex})$$

$$= e^{-\zeta\omega_0 t} \; B_1 \cos \; \omega_0\sqrt{1-\zeta^2}\,t \; + B_2 \sin \; \omega_0\sqrt{1-\zeta^2}\,t$$

$$= Ce^{-\zeta\omega_0 t} \cos \; \omega_0\sqrt{1-\zeta^2}\,t + \phi$$

Particular, Complete Response

- For unit step input, particular response = constant.

$$v_{o,p}(t) = K$$

- Complete response (overdamped):

$$v_o(t) = v_{o,p}(t) + v_{o,h}(t)$$

$$= K + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- Complete response (underdamped):

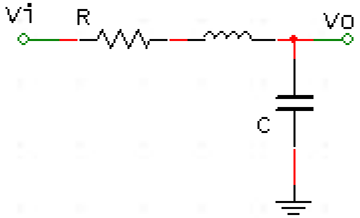
$$v_o(t) = v_{o,p}(t) + v_{o,h}(t)$$

$$= K + Ce^{-\zeta\omega_0 t} \cos \ \omega_0 \sqrt{1-\zeta^2}\, t + \phi$$

- Note: K is determined by input and differential equation. Constants of homogeneous response are determined by initial conditions.

Initial Conditions

For this circuit, and step input:



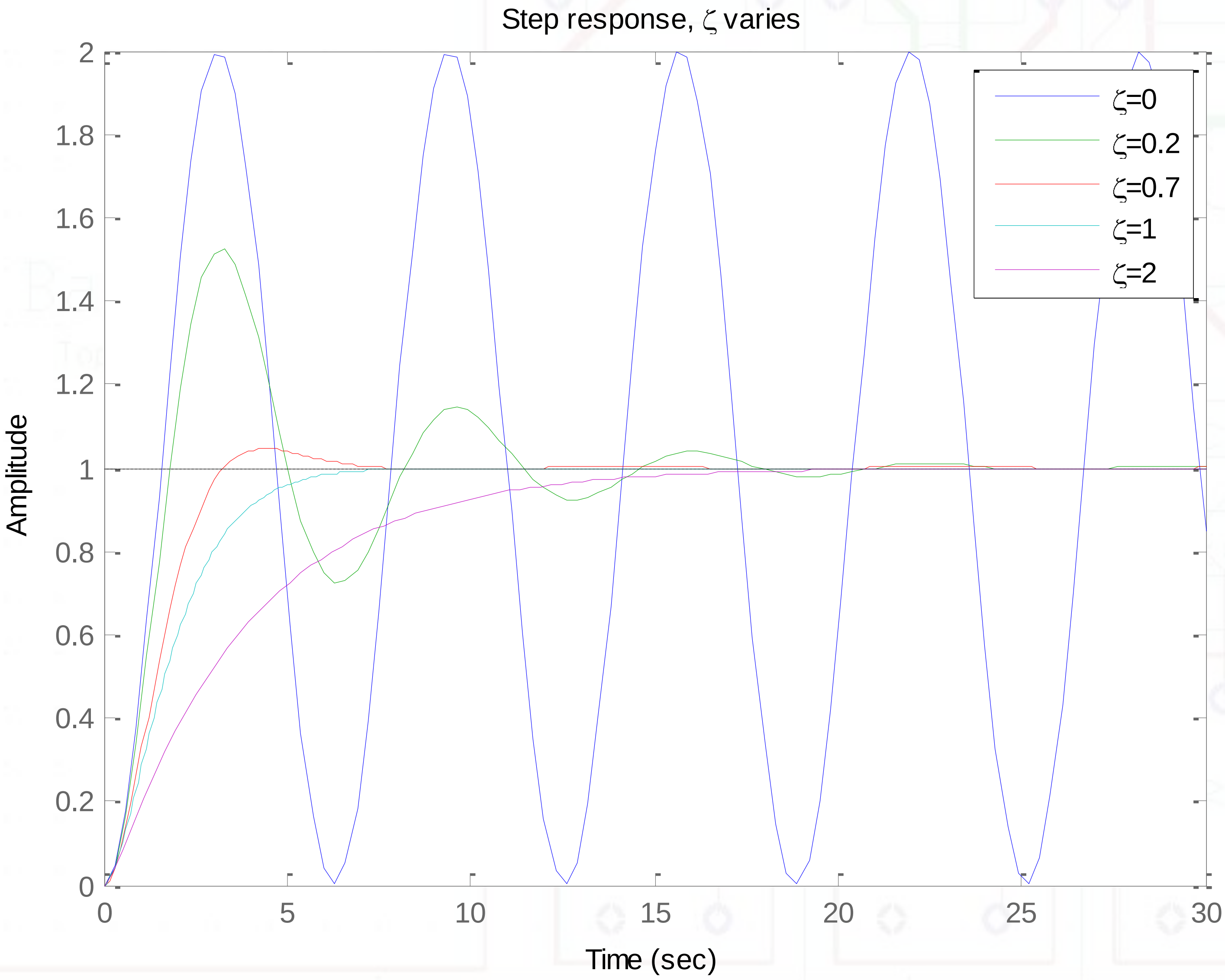
$$\frac{d^2 v_o(t)}{dt^2} + \frac{R}{L}\frac{dv_o(t)}{dt} + \frac{1}{LC} v_o(t) = \frac{1}{LC} v_i(t)$$

$$v_o(0^+) = \ldots ? \qquad v_o(0^+) = 0$$

$$\left.\frac{dv_0(t)}{dt}\right|_{t=0^+} = \ldots ? \qquad \left.\frac{dv_0(t)}{dt}\right|_{t=0^+} = 0$$

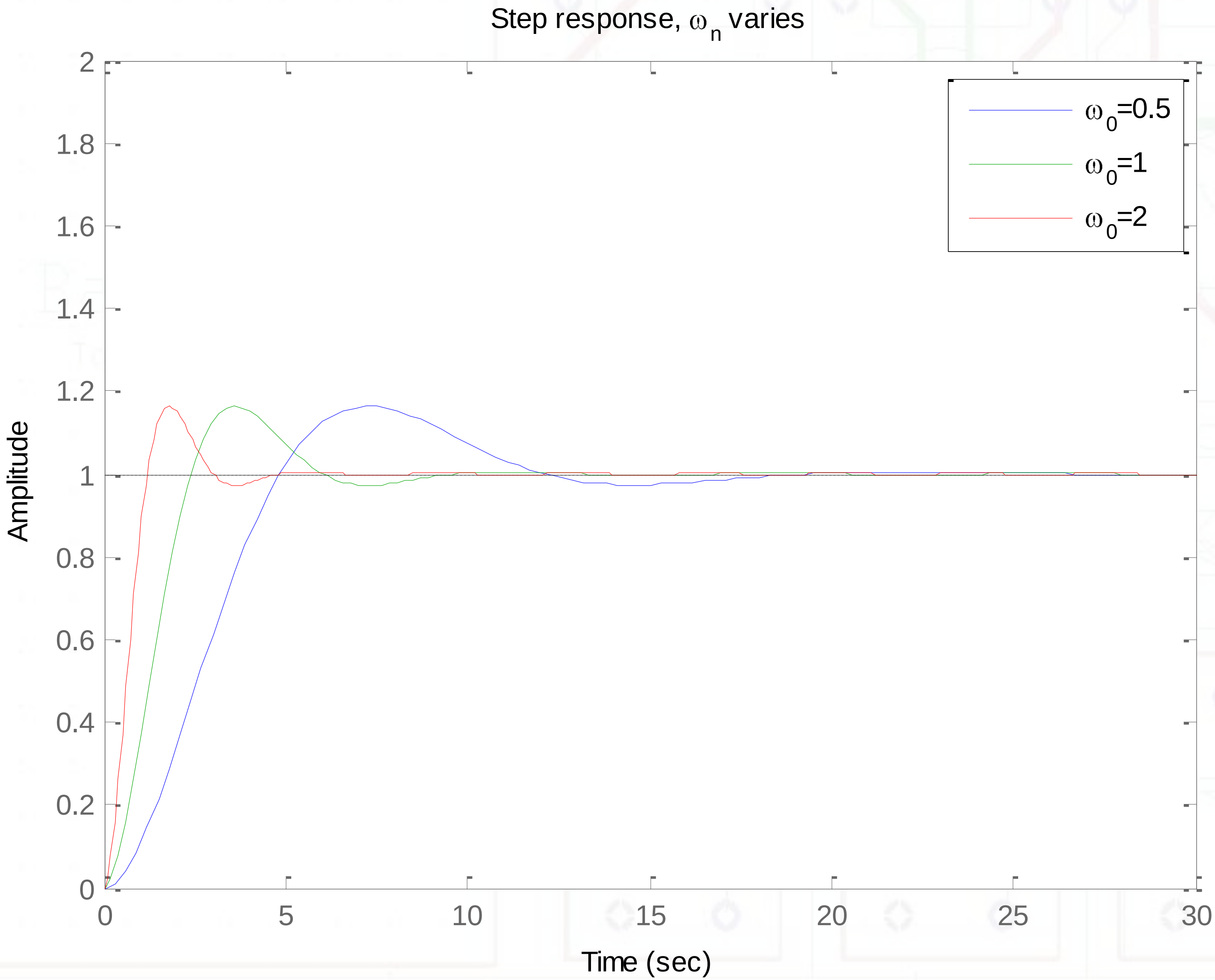
Effect of ζ

Increasing ζ denotes increasing damping (ζ determines nature of response)



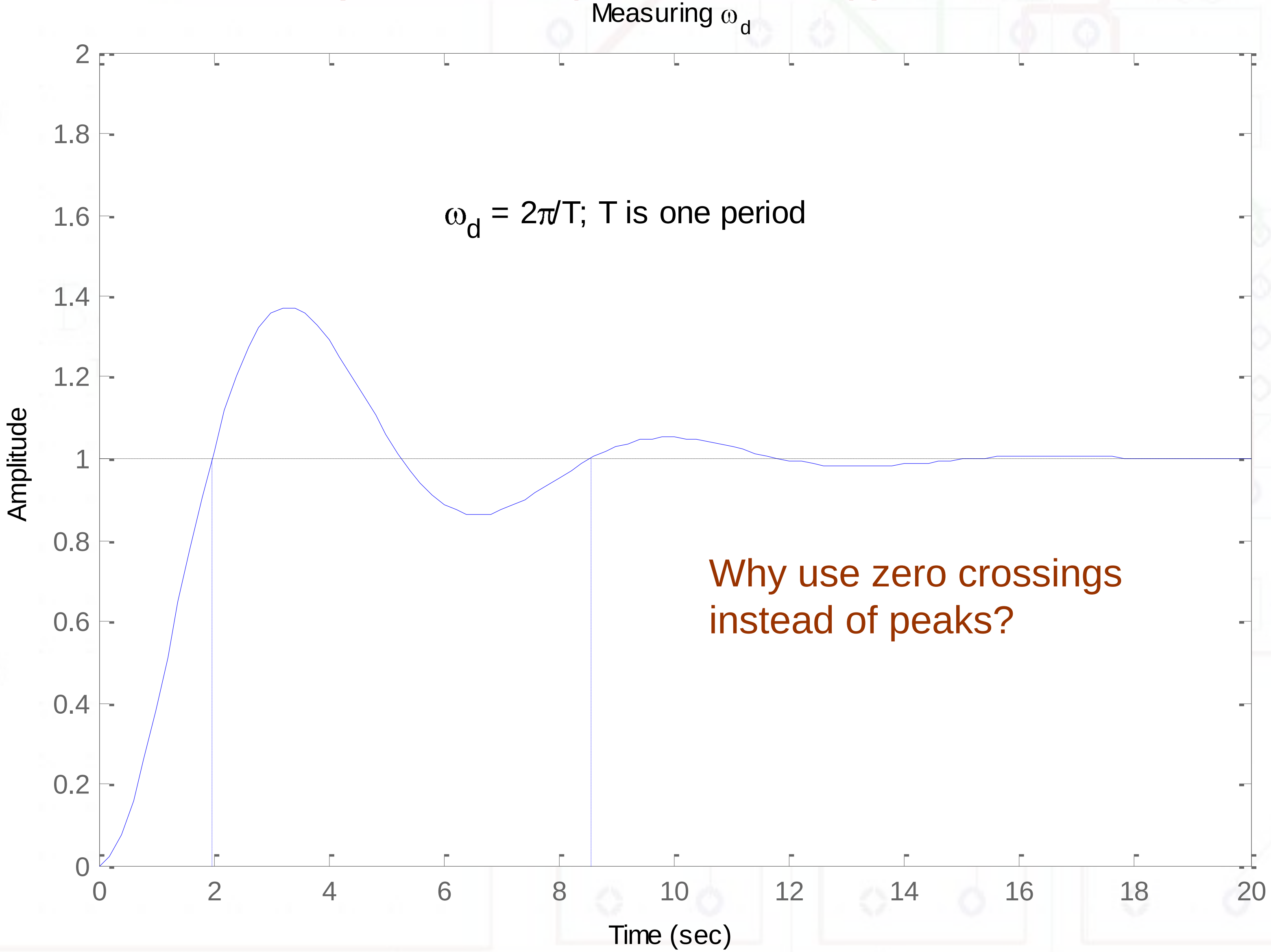
Effect of ω_0

Increasing ω_0 denotes increasing speed (ω_0 determines speed of response)



In-Lab Measurements (ω_d)

(underdamped case only)



$\omega_d = 2\pi/T$; T is one period

Why use zero crossings instead of peaks?

In-Lab Measurements (α)

(underdamped case only)

$$y1 = a + b \cdot e^{-\alpha \cdot t1}$$

$$y2 = a + b \cdot e^{-\alpha \cdot t2}$$

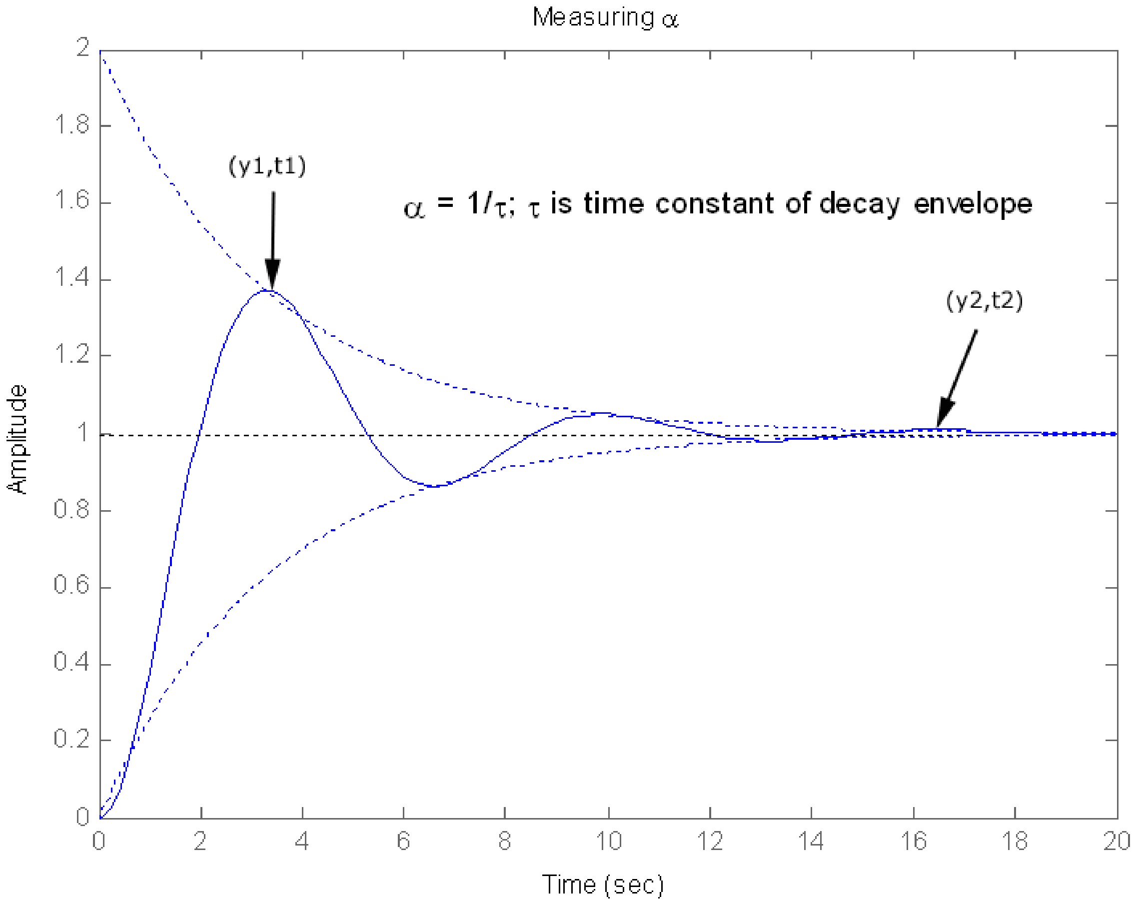
$$\frac{y1 - a}{y2 - a} = \frac{b \cdot e^{-\alpha \cdot t1}}{b \cdot e^{-\alpha \cdot t2}}$$

$$= e^{-\alpha \cdot \ t1 - t2}$$

$$\log\left(\frac{y1 - a}{y2 - a}\right) = -\alpha \cdot \ \ t1 - t2$$

This is called the logarithmic decrement.

$$\alpha = \frac{\log\left(\frac{y1 - a}{y2 - a}\right)}{t2 - t1}$$



Note: we could also do a curve fit for more accuracy (and use all the data).

Switching parameters

(underdamped case only)

$$\alpha = \zeta\omega_0$$

$$\omega_d = \omega_0\sqrt{1-\zeta^2}$$

$$\alpha^2 = \zeta^2\omega_0^2$$

$$\omega_d^2 = \omega_0^2 \;\; 1-\zeta^2$$

$$\omega_d^2 = \omega_0^2 - \zeta^2\omega_0^2$$

$$= \omega_0^2 - \alpha^2$$

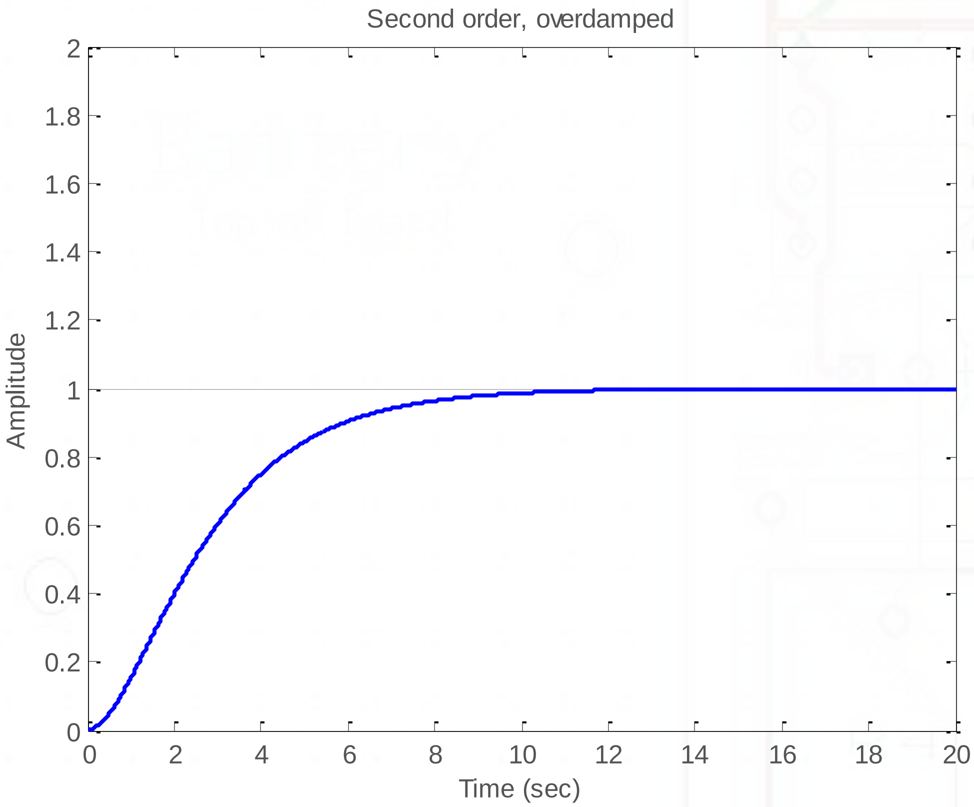
$$\omega_0 = \sqrt{\omega_d^2 + \alpha^2}$$

$$\zeta = \frac{\alpha}{\omega_0}$$

If you do a curve fit, you can solve directly for ζ and ω_n.

The Overdamped Case

For overdamped (and critically damped) case, there is no easy way to find the two time constants from the experimental data unless you do a curve fit.



$$y(t) = A_1 e^{-t/\tau 1} + A_2 e^{-t/\tau 2} + 1$$

A curve fit is made difficult because the unknown coefficients (A_1, A_2, τ_1, τ_2) are hard to make initial guesses for by looking at the graph – but you can use theoretical values as your initial guesses.

If you do a curve fit, you can solve directly for ζ and ω_n.