Polygonal billiards and interval maps

I am primarily interested in questions about billiards in all of their many forms. The study of billiards has exploded in popularity recently; two of the 2014 Fields Medalists (Mirzakhani and Avila) work in related areas. Billiards exhibit beautiful structure, and they are perfect for student research since there is a low barrier to entry.

I have worked on several different kinds of billiards problems; the difference is the rule about what happens when a linear trajectory hits an edge (see Figure 1).

- **Classic polygonal inner billiards**, where a trajectory hitting an edge bounces off and the angle of incidence equals the angle of reflection (red). I wrote about billiards in the square in great detail in my expository book [D17], I studied reconstructing a table from its bounce sequences in [CCDLO18], and I’ve studied periodic trajectories in the regular pentagon in [DFT11, DL18]; see §1.

- **Tiling billiards**, where a trajectory hitting an edge of a planar tiling reflects across the edge (yellow). This is a new field that I’ve largely developed, which has potential industrial applications, and which I’ve studied with students and colleagues for several families of tilings in [DDRS18, DH18, BDFI18]; see §2.

- **Translation surfaces**, where a trajectory hitting an edge emerges from a parallel, oppositely-identified edge (green). I studied these in my Ph.D. thesis and the work that followed, in [D13, D14, DPU18]; see §3.

- **Polyhedra**, where a trajectory hitting an edge keeps going as though the two faces were coplanar (blue). I studied these in [DDoTY17]; see §4.

Figure 1: The reflection rule when hitting an edge in (a) inner billiards, §1 (b) tiling billiards, §2 (c) polygonal surfaces, §3 (d) polyhedra, §4

I also have two current projects about interval maps:

- **Piecewise-linear interval maps**, studying the properties of the set of Galois conjugates of roots of a special class of such maps; see §5.

- **Measure-preserving transformations of the interval**, showing that most such transformations are not interval exchange transformations; see §6.


1 Billiards and periodicity

The simplest case of polygonal billiards is a ball bouncing around in a square, which has been well understood for at least a hundred years:

**Theorem.** A billiard path in a square, in the direction of slope $p/q$ in lowest terms, is periodic with period $2(p + q)$.

The idea is that the vertex-to-vertex vector in the direction of slope $p/q$ is $[q, p]$, and a trajectory hits $p + q$ edges in creating that vector. Since at least one of $p$ or $q$ is odd, we double the path to fix the orientation, so we hit $2(p + q)$ edges.

The more general question is: For a billiard in a polygon, given a periodic direction, what is the period of a trajectory in that direction? As of a few years ago, we could only answer this question for rectangular tables. Thanks to my work with Dmitry Fuchs and Sergei Tabachnikov [DFT11], and with Samuel Lelièvre [DL18], we now know the answer for an additional shape: the regular pentagon. Some examples of periodic paths in the regular pentagon are below; we made the pictures using Sage.

![Figure 2: Some periodic billiard paths on the regular pentagon.](image)

Our result shows that the answer for the pentagon is “the same” as for the square, as long as we define the quantities analogous to the slope $p/q$ correctly. For this work, we use the golden $L$, a right-angled surface that is affinely equivalent to the double pentagon surface, and we put a tree structure on the infinite set of periodic directions.

**Theorem.** (Davis and Lelièvre [DL18]). In each periodic direction, there are two periodic trajectories, whose corresponding direction in the golden $L$ has long saddle connection vector $[a + b \phi, c + d \phi]$. The short trajectory in this direction has holonomy vector $v = [a + b \phi, c + d \phi]$ in the golden $L$ and combinatorial period $2(a + b + c + d)$ in the double pentagon, and the long trajectory in this direction has holonomy vector $\phi v$ in the golden $L$ and combinatorial period $2(a + 2b + c + 2d)$ in the double pentagon.

Here $\phi = \frac{1 + \sqrt{5}}{2}$ is the familiar golden ratio.

**Outreach.** Because the periodic billiard trajectories (see examples above) are very beautiful, I have been making them into jewelry. I have given these “beautiful objects from research mathematics” to hundreds of people, to popularize research mathematics among women and members of other under-represented groups in mathematics.
Future work

Given that we only really understand periodic trajectories for the regular pentagon and the simplest of other shapes, the pressing need now is to extend our work to the other regular and rational polygons, and understand the structure of their (countable, dense) set of periodic directions. Planned extensions of our work are as follows:

Regular polygons. Samuel Lelièvre and I plan to extend our methods to the rest of the regular polygons (regular heptagon, octagon, and in general), to completely describe the structure of their periodic paths and develop the theory in general. We expect that this work will yield beautiful pictures like Figures 2, with novel properties that we cannot yet anticipate.

Non-equidistribution. Regular polygons exhibit optimal dynamics: every billiard path is either periodic or equidistributed. We found some examples of non-equidistributed long periodic trajectories, which were highly unexpected, as they violate the spirit of this dichotomy (albeit not literally, since they are all periodic). Fields Medalist Curtis McMullen is currently studying this phenomenon that we discovered; see his November 2018 Coxeter lecture at the Fields Institute, especially 55:30-60:00.

Extending our understanding of the pentagon case. While our paper [DL18] answered the basic questions about directions, saddle connection vectors and combinatorial periods of periodic directions on the regular pentagon, much remains to be done. Using our Sage program, we collected computational evidence toward answering questions about the combinatorics of the bi-infinite sequences of edges that a billiard ball hits; we plan to prove the resulting conjectures.

Undergraduate research. My undergraduate thesis student Megumi Asada (at Williams College in 2016–2017) proved theorems explaining the structure of periodic directions on the equilateral triangle, and also did some work on the regular hexagon. I hope to have future research students who will continue studying accessible parts of this problem. Many mathematicians who study billiards have written packages in Sage to do the specific calculations needed in the field, so many resources already exist for a student with an interest or background in programming.

2 Tiling billiards and negative refraction

Over the past five years I have developed a new area of billiards research, with several groups of undergraduates. The system, which we call tiling billiards, is motivated by a discovery in physics about 15 years ago of metamaterials, which have a negative index of refraction, and thus refract “backwards” [SSS01, SPW04]. Metamaterials have many applications, for example to creating a perfect lens or an invisibility shield. We study how trajectories of light would refract through a tiling of the plane that alternates standard materials and metamaterials, with equal and opposite indices of
refraction, so that angles are preserved at bounces. Several examples are in Figure 3.

![Figure 3: Two tiling billiards trajectories on the isosceles right triangle tiling.](image)

With then-undergraduates Kelsey DiPietro, Jenny Rustad and Alex St Laurent at the Summer@ICERM REU, we explored tiling billiards on three classes of tilings, proving several results and making many conjectures about periodicity and stability.

In subsequent work with W. Patrick Hooper, we studied the billiard flow on the trihexagonal tiling, where an equilateral triangle and a regular hexagon meet at every edge, by relating it to an infinite translation surface that is a cover of the torus \([DH18]\). We used the surface to show that, for this tiling, dense behavior is general:

**Theorem.** \([DH18]\) For almost every initial point and direction, a trajectory with this initial position and direction is dense in all of the plane except for a periodic family of triangular open sets.

In my subsequent undergraduate research group at the SMALL REU at Williams College in 2016 with then-undergraduates Paul Baird-Smith, Elijah Fromm and Sumun Iyer, we extended the results of \([DDRS18]\) for triangle tilings, showing that the system can be described by an interval exchange transformation (see Figure 6 and §6) and that fractal-like trajectories can arise \([BDFI18]\). We proved the following powerful result about trajectories on triangle tilings:

**Theorem.** \([BDFI18]\) The movement of a trajectory whose chord subtends angle \(\tau\) in a triangle tiling with angles \(\alpha, \beta, \gamma\) is described by the orbit of a point under the tiling billiards IET, which is a circle exchange transformation: the unit circle is cut into intervals of length \(2\alpha, 2\beta, 2\gamma\), each interval is flipped in place, and the circle is rotated by \(\tau\).

This connects tiling billiards to interval exchange transformations, about which there is a substantial literature and much interest. We also gave examples of certain triangle tilings where the trajectories approach fractals, and subsequently Pascal Hubert and Olga Paris-Romaskevich proved their fractal-like structure.
Future work

A multitude of tilings remain to be studied; we will begin with the following:

*Penrose tilings.* Pat Hooper and I are currently working on tiling billiards on the aperiodic Penrose tiling, where preliminary evidence suggests that, surprisingly, almost every trajectory is periodic.

*The wind-tree model.* In the past 10 years there has been a lot of interest in the *wind-tree model* of billiards, where there are congruent obstacles (“trees”) at every lattice point in a given lattice, and a ball (“wind”) bounces around in their complement. With Ferrán Valdez and Olga Romaskevich, I am working on studying this system in the tiling billiards model, where the obstacles are made of the metamaterial.

*Undergraduate research.* Almost all of the work on this question has been done by undergraduates. It is a wonderful student research topic, because the students can understand the problem within a few minutes, and begin coming up with observations and conjectures on the first day. We have a Java program that models the system, and students interested in programming can easily add additional tilings, and other features to the program specific to the questions they wish to explore.

3 Trajectories on polygonal surfaces

In my Ph.D. thesis, I studied surfaces made by identifying opposite parallel edges of polygons, called *translation surfaces*, the canonical example of which is the square torus. This is related to billiards because given a polygonal billiard table with rational angles, we can *unfold* the table across its edges until every edge has an oppositely oriented parallel match, which we then identify to yield a translation surface. Given a trajectory (usually a geodesic) on such a surface, we can write down the bi-infinite sequence of edges that the sequence crosses, called the associated *cutting sequence*.

One goal of this work is to be able to look at a bi-infinite string of edge labels, and determine if it corresponds to a geodesic on the surface. One way to do this is to use *derivation*, a process that is finite for periodic directions, infinite for aperiodic directions, and fails for sequences that are not cutting sequences − constructs the continued fraction expansion of the slope of the trajectory for the square, and is analogous on other surfaces. Caroline Series developed this structure for the square in [Se85a, Se85b, Se91], and I explore it in depth in my expository book [D17]. The idea is to use a certain *shear* automorphism of the surface to obtain a simpler trajectory from a given starting trajectory.

I characterized the effect of this derivation process for surfaces that are more complicated than the square but still have certain “lattice” symmetries: the double regular pentagon and regular $2n+1$-gons in general, and Bouw-Möller surfaces, which are a large family of surfaces of which the regular polygon surfaces are a subset [BM06, Ho13].
Theorem. [D13] For the double regular $n$-gon, the result of applying the shear $\begin{bmatrix} -1 & \frac{2 \cot \pi/n}{2} \\ 0 & 1 \end{bmatrix}$, as induced on the cutting sequence corresponding to a linear trajectory, is to keep only the edge labels with the same symbol on each side.\footnote{1}

In [D14], I determined the derivation rule for certain trajectories on the Bouw-Möller surfaces. Then in joint work with Corinna Ulcigrai and Irene Pasquinelli, we were able to treat all cutting sequences on Bouw-Möller surfaces, and determine a derivation rule for them [DPU18]. The key was the structure of transition diagrams, which describe how linear trajectories cut through edges of the polygons. Using the transition diagrams, we gave the derivation rule for all trajectories, which uses the relationship between the $(m, n)$ and $(n, m)$-indexed Bouw-Möller surface, and reduces to the above Theorem in the case of the double polygons.

4 Geodesics on polyhedra

Sometimes people ask, “What about billiards in three dimensions?” One way to take two-dimensional dynamics into the third dimension is to wrap a linear trajectory around a polyhedron. A straight path on the cube, for example, is essentially a straight path on the square grid, except that three squares come together at a vertex instead of four, and this makes things interesting as the path wraps around and around the cube. The question I have sought to answer is: Given two points $A$ and $B$ on a polyhedron, which directions can we go from $A$ to get to $B$? See Figure 4.

Figure 4: A path between two vertices of the cube. Figure by Joseph O’Rourke.

In joint work with Victor Dods, Cindy Traub and Jed Yang, we studied vertex-to-vertex paths on the regular tetrahedron and the cube, with many results, including:

Theorem. [DDoTY17] Given a vertex on the cube, there are other vertices that are a distance $1$, $\sqrt{2}$ and $\sqrt{3}$ away. The (countably infinite) number of paths to vertices of each type are in a 4:3:6 ratio.

\footnote{1}I made an award-winning video that explains the double pentagon surface, and this result, using colors and dance: http://vimeo.com/47049144
To do this, we used the structure of the Stern-Brocot tree on the rational numbers, and computer programming to perform large calculations. Dmitry Fuchs and Ekat-
erina Fuchs have related results for closed trajectories on regular polyhedra, using different methods [FF07], and D. Fuchs continues to work in this area.

**Future work and undergraduate research opportunities**

We would like to be able to answer the question italicized above for every polyhedron, and for every pair of points $A$ and $B$. The answer is only known for regular polyhedra, and even then only for vertex-to-vertex paths (except on the tetrahedron where everything is easy). Jayadev Athreya, David Aucilino and Pat Hooper have recently posted two papers about vertex-to-vertex paths on regular polyhedra, extending our work by using the technology of flat surfaces [AA18, AAH18]. Ideally, we would work to answer this question for any pair of points on any polyhedron. These problems are simple to explain, and they lend themselves well to computer experimentation, so they are also perfect for student projects.

5 Tent maps of the interval

In his last paper, unfinished at the time of his death, William Thurston studied piecewise-linear maps of the unit interval [T14]. Of particular interest are tent maps as shown in Figure 5(a), functions of the form

$$f(x) = \begin{cases} 
\beta x, & 0 \leq x \leq 1/\beta \\
2 - \beta x, & 1/\beta \leq x \leq 1 
\end{cases},$$

for some $1 < \beta < 2$. We are interested in the slopes $\beta$ for which the orbit of 1 under the associated tent map is periodic. All such $\beta$ are algebraic, so we can consider the set of all of their Galois conjugates. This is the Thurston set, shown in Figure 5(b), which is visually striking and exhibits fascinating fractal-like structures.

Figure 5: (a) A tent map with $\beta = 1.4$. (b) The Thurston set (picture by Thurston).
In joint work with Harrison Bray, Kathryn Lindsey and Chenxi Wu, we prove several statements that Thurston made in his paper about the three-dimensional set he called the Master Teapot: the set of points \((x, y, \beta)\), where \((x, y)\) is in the Thurston set and \(\beta\) is the slope of the associated tent map; for example:

**Theorem.** [BDLW18] The Master Teapot is connected. Furthermore, its intersection with the solid cylinder \(D \times [1, 2]\) is path-connected.

### 6 Measure-preserving transformations of the interval

As mentioned in §2, interval exchange transformations (IETs) are of great interest, and are related to billiards in polygons. An IET is a measure-preserving transformation of the interval, where the interval is cut up into (usually finitely many) pieces, which are then re-ordered. See Figure 6 for an example.

![Interval Exchange Transformation](image)

Figure 6: An interval exchange transformation, which cuts the unit interval (top) into three pieces and rearranges them (bottom) by translation.

IETs are far from the only measure-preserving transformations of the interval. In joint work with Jon Chaika, we are working to show that, in fact, most measure-preserving transformations are not IETs:

**Conjecture.** [CD18] A typical measure-preserving transformation is not isomorphic to any interval exchange transformation.

We have proven this in the case of IETs that satisfy the Keane condition, which is essentially that all of the images of a given point are disjoint, and we are working to prove it in general.
References


Dmitry Fuchs and Ekaterina Fuchs, *Closed geodesics on regular polyhedra*, *Moscow Mathematical Journal* 01/2007; 7(2).


