Mathematician spotlight: Kwadwo Antwi-Fordjour, Earlham College
- modeling head re-growth of hydra using differential equations
- speaking TODAY 4:30 pm in SCI 181 (VAP job candidate)

Last time: To find absolute extrema of a function on a bounded region, you have to check for critical points on interior, critical pts of boundary curve(s), and corners.

Today & Monday: Absolute extrema on a constraint curve using Lagrange multipliers.

Example. (review using strategy from last time) Find absolute extrema of \( f(x,y) = xy \) over the closed unit disk \( x^2 + y^2 \leq 1 \).

1. Find critical points on interior by setting \( \nabla f = 0 \):
   \[
   \nabla f = \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \Rightarrow (0,0) \text{ is the only critical point.}
   \]

2. Find critical points of \( f \) on the boundary curve:
   boundary curve \( x^2 + y^2 = 1 \)
   is unit circle: \( x = \cos \theta, y = \sin \theta \)
   \( f(x,y) = xy \)
   \( f(\theta) = \cos \theta \cdot \sin \theta \)

Now check value of \( f \) at these points:
\[
\begin{align*}
  f(0,0) &= 0 \hfill \text{neither a max nor a min} \\
  f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) &= f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{1}{2} \hfill \text{two equal maxima} \\
  f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) &= f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{1}{2} \hfill \text{two equal minima}
\end{align*}
\]

Lagrange's great idea: a new strategy for finding critical points of \( f \) on the boundary.
- This is a topographical (elevation) map of a piece of land.
- Draw a lumpy blob outline. This is the fence that encloses your herd of sheep.
- Find the points of highest & lowest elevation of the fence. How did you find them?

Lagrange's idea: At a critical point of the function \( f(x,y) \) on the constraint \( g(x,y) = k \)
- the boundary curve is tangent to a level curve of \( f \)
- \( \nabla f \) and \( \nabla g \) point in the same direction: they are multiples of each other.

So, to optimize \( f(x,y) \) subject to constraint \( g(x,y) = k \), solve the Lagrange multipliers equation \( \nabla f \left( x_0, y_0 \right) = \lambda \cdot \nabla g \left( x_0, y_0 \right) \), where \( \lambda \) is some constant.
Example: Find absolute extrema of \( f(xy) = xy \) on the unit circle. (same as before)

We wish to maximize/minimize the function \( f(xy) = \) 
subject to the constraint \( g(xy) = \) 

By the Lagrange multiplier equation,

\[
\nabla f = \lambda \nabla g \Rightarrow \begin{bmatrix} y_2 \\ x_2 \\ xy_x \end{bmatrix} = \lambda \begin{bmatrix} 2y_x \\ 2x_y \\ y_2 \end{bmatrix} \Rightarrow \begin{cases} y_2 = \lambda x_2 \\ x_2 = \lambda y_2 \\ xy_x = \lambda \end{cases}
\]

\[
\Rightarrow \begin{cases} x_2 = \lambda \\ y_2 = \lambda x_2 \Rightarrow y_2 = \frac{\lambda^2}{2} \\ y_2 = \frac{x_2}{\lambda} \Rightarrow x_2 = \frac{\lambda}{2} \end{cases}
\]

\[
\frac{x_2}{\lambda} = \frac{\lambda}{2} \Rightarrow 2 = y
\]

3 eqns, 3 variables

Finally, plug this into the constraint equation to solve for points:

\[
y = \pm x \text{ and } x^2 + y^2 = 1 \Rightarrow (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \text{ and } (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \text{ maxes as before.}
\]

\[
y = -x \text{ and } x^2 + y^2 = 1 \Rightarrow (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \text{ and } (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \text{ mins}
\]

Example: Find the largest possible product of three numbers whose sum is 100.

\( \Rightarrow \) not possible because

How about: Find the largest possible product of three **positive** numbers whose sum is 100.

Let the three numbers be \( x, y, z \). We seek to maximize \( f(x, y, z) = \) 
under the constraint \( g(x, y, z) = \) 

By the Lagrange multipliers equation,

\[
\nabla f = \lambda \nabla g \Rightarrow \begin{bmatrix} y_2 \\ x_2 \\ xy_x \end{bmatrix} = \lambda \begin{bmatrix} 2y_x \\ 2x_y \\ y_2 \end{bmatrix} \Rightarrow \begin{cases} y_2 = \lambda x_2 \\ x_2 = \lambda y_2 \\ xy_x = \lambda \end{cases}
\]

\[
\Rightarrow \begin{cases} x_2 = \lambda \\ y_2 = \lambda x_2 \Rightarrow y_2 = \frac{\lambda^2}{2} \\ y_2 = \frac{x_2}{\lambda} \Rightarrow x_2 = \frac{\lambda}{2} \end{cases}
\]

4 eqns, 4 variables

Now plug into our constraint: \( x + y + z = 100 \Rightarrow x + x + x = 100 \Rightarrow x = \frac{100}{3} \Rightarrow \) numbers are all \( \frac{100}{3} \).

How do we know its a max, not a min? \( f(\frac{100}{3}, \frac{100}{3}; \frac{100}{3}) \leq 37037 \)

\( f(33, 33, 33) = 37026 \Rightarrow \) function value at nearby point is lower, so it's a max.

Example: Find the largest volume of a rectangular box whose length, width, height sum to 100.

\( \Rightarrow \) Did that!

How about: Find the largest volume of an open-top rectangular box made of 12 ft\(^3\) of cardboard.

We wish to maximize \( f(x, y, z) = \) 
subject to constraint \( g(x, y, z) = \) 

so \( \nabla f = \lambda \nabla g \Rightarrow \begin{bmatrix} y_2 \\ x_2 \\ xy_x \end{bmatrix} = \lambda \begin{bmatrix} y + 2z \\ x + 2z \\ 2x + 2y \end{bmatrix} \)

... etc.