Interval exchange transformations from tiling billiards

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Elijah Fromm, Sumun Iyer and Paul Baird-Smith
Standard (Inner) Billiards:

trajectory is reflected against the edge of the table, preserving angles
In the square billiard table, a trajectory with:

**rational** slope \( p/q \)
is periodic, with period \( 2(p+q) \)

**irrational** slope is
dense, filling in the entire table.
In the square billiard table, a trajectory with:

**rational** slope $p/q$ is periodic, with period $2(p+q)$

**irrational** slope is dense, filling in the entire table.
In the square billiard table, a trajectory with:

- **rational** slope $p/q$ is periodic, with period $2(p+q)$
- **irrational** slope is dense, filling in the entire table.
In the square billiard table, a trajectory with:

**rational** slope $p/q$

is periodic, with period $2(p+q)$

and

**irrational** slope is dense, filling in the entire table.
In the square billiard table, a trajectory with:

- **rational** slope $p/q$ is periodic, with period $2(p+q)$
- **irrational** slope is dense, filling in the entire table.
Standard (Inner) Billiards:

- A trajectory is **dense** with probability 1
- Behavior is **unstable**
Snell’s Law:
governs the refraction of a beam of light passing from one material to another

\[ \frac{\sin \theta_1}{\sin \theta_2} = \frac{k_1}{k_2} \]

standard material, positive index of refraction \( k_1 \)

standard material, positive index of refraction \( k_2 \)
Snell’s Law:

governs the refraction of a beam of light passing from one material to another

\[
\frac{\sin \theta_1}{\sin \theta_2} = \frac{k_1}{k_2}
\]

standard material, positive index of refraction \( k_1 \)

metamaterial, negative index of refraction \( k_2 \)
Snell’s Law: governing the refraction of a beam of light passing from one material to another.

\[
\frac{\sin \theta_1}{\sin \theta_2} = \frac{k_1}{k_2}
\]

materials with equal and opposite indices of refraction.
Applications:
1. Invisibility cloak
2. Perfect lens

A cloak made of a negative-index metamaterial can bend radiation around an object inside it, making that object seem invisible.
Applications:
1. Invisibility cloak
2. Perfect lens
Tiling Billiards:
A dynamical system where light refracts through a planar tiling by materials with equal and opposite alternating indices of refraction
Standard Billiards:

trajectory is reflected **against** the edge of the table, preserving angles

Tiling Billiards:

trajectory is reflected **across** each edge of the tiling, preserving angles
Standard Billiards:

trajectory is reflected **against** the edge of the table, preserving angles

Tiling Billiards:

trajectory is reflected **across** each edge of the tiling, preserving angles
Standard (Inner) Billiards:

- A trajectory is **dense** with probability 1
- Behavior is **unstable**

Tiling Billiards on triangle tilings:

- Most trajectories are **periodic**
- Trajectories are very **stable**

**Burning question:** What causes periodicity and stability in tiling billiards on triangle tilings?
Lemmas (SMALL ’16):

- Trajectory folds to a single line
- Folded triangles share a circumcircle
- All blue triangles end up on the same side
**Insight** (SMALL ’16):

- Fix trajectory; triangle moves
- Keep track of favorite vertex
- This yields a 1-dimensional system, in fact an Interval Exchange Transformation (IET).
Let $X$ be the location of the identified vertex, and $\tau$ the angle subtended by the trajectory chord.

Our IET is defined by:

$$X' = \begin{cases} 
\tau + 2\beta - X & \text{if } 0 < X < 2\beta \\
\tau + 2\beta - 2\gamma - X & \text{if } 2\beta < X < 2\beta + 2\gamma \\
\tau - 2\gamma - X & \text{if } 2\beta + 2\gamma < X < 2\pi,
\end{cases}$$

an orientation-reversing IET.
Tiling Billiards on triangle tilings:

- Give a 3-IET on the circle
- Interval lengths: $2\alpha, 2\beta, 2\gamma$
- Shift transformations: based on $\alpha, \beta, \gamma, \tau$
- Are orientation-reversing (“fully flipped”)
Why flipped IETs are stable & periodic
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Why flipped IETs are stable & periodic
Why flipped IETs are stable & periodic

period 2

period 2

period 6
Why flipped IETs are stable & periodic

Everything is flipped periodic, every point is stable, periods of form $4n+2$
Standard (Inner) Billiards:

- A trajectory is dense with probability 1
- Behavior is unstable

Tiling Billiards on triangle tilings:

- Most trajectories are periodic
- Trajectories are very stable

Burning question: What causes periodicity and stability in tiling billiards on triangle tilings?

They correspond to fully flipped IETs.
Comparison to non-flipped IETs

If $|AB|$ and $|C|$ are irrationally related, every point is aperiodic (rotation).
Tiling billiards corresponds to orientation-reversing 3-IET

Idea:

• Use the **square** of the 3-IET
• Get an orientation-**preserving** 6*-IET
• Stack all of them into a PET
Tiling billiards PET: stack of IETs

\[
\begin{bmatrix}
2\pi \\
\tau \\
\end{bmatrix}
\]

- **BC** \((X' = X + 2\alpha)\)
- **BA** \((-2\gamma)\)
- **AB** \((+2\gamma)\)
- **AC** \((-2\beta)\)
- **CA** \((+2\beta)\)
- **CB** \((-2\alpha)\)

IET

IET
Tiling billiards PET: stack of IETs

Diagram Showing:
- Region AC
- Region CA
- Region BC
- Region CB
- Region BA
- Region AB

Axes:
- 0
- \( \tau \)
- \( X \)
- \( 2\beta \)
- \( 2\beta + 2\alpha \)
- \( 2\pi \)
Visit the zoo:
Billiard trajectories on triangle tilings
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Billiard trajectories on triangle tilings
The Rauzy fractal as a billiard trajectory
The Rauzy fractal as a billiard trajectory!
Future work:

- Show that we actually get fractals as the limit of billiard trajectories
- Completely understand fully flipped IETs (service to community)
- Other tilings!