The problems in this text

The method of instruction used with these problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. Most of the beginning of the course (and some of the later) is based on *Real Analysis* by Frank Morgan. Most of the end of the course (and some of the earlier) is based on Aimee Johnson’s lecture notes and worksheets, which in turn are based on *Introduction to Analysis* by Maxwell Rosenlicht. Some of the problems were written by Diana Davis specifically for this course. If you create your own text using these problems, please give appropriate attribution, as I am doing here.

About the course

This course met three mornings a week for 50 minutes, for which the homework was the numbered pages, one page per class, for a total of 42 “discussion sessions.” We also met once a week in the afternoon for a 90-minute “problem session,” during which the students worked in groups on the problem pages whose numbers begin with P. They typed up (in \LaTeX) and handed in solutions to four of these problems later in the week. The course was designed to teach students how to write a proof, in addition to the content of analysis.

To the Student

**Contents:** As you work through this book, you will discover that the various topics of real analysis have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on sequences of functions. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important ideas. Key words are defined in the problems, where they appear italicized.

**Your homework:** Each page of this book contains the homework assignment for one night. The first day of class, we will work on the problems on page 1, and your homework is page 2; on the second day of class, we will discuss the problems on page 2, and your homework will be page 3, and so on for each day of the semester. You should plan to spend two to three hours solving problems for each class meeting.

**Comments on problem-solving:** You should approach each problem as an exploration. Draw a picture whenever appropriate. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day. Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you’re probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.
Help me help you!

Please be patient with me, as I try to be patient with you. I have spent a long time working on this set of problems, thinking hard about each problem and how they all connect and build the ideas, step by step. I’ve done my best, so trust me, you have the tools to solve the problems! On the other hand, I may have made some mistakes, and for that, I apologize in advance. Just remember that we are all in this together. Our goal is for each student to learn the ideas and skills of Real Analysis, really learn them — and along the way I will learn new things, too. That’s the beauty of this teaching and learning method, that it recognizes the humanity in each of us, and allows us to communicate authentically, person to person.

One way of describing this method is “the student bears the laboring oar.” This is a metaphor: You are rowing the boat; you are not merely along for the ride. You do the work, and in this way you do the learning. The next page gives some ideas for ways that you can do this work of moving the “boat,” which is our class and your learning, forward.

You might wonder, what is my job as your teacher? Part of my job is to give you good problems to think about, which are in this book. During class, my job is to help you learn to talk about math with each other, and help you build a set of problem-solving strategies. At the beginning, I will give you lots of pointers, and as you improve your skills I won’t need to help as much. I might say things like

- “Please go up to the board and write down what you’re saying.”
- “Get some colored chalk and add that to the picture on the board.”
- “You were confused before, and now it sounds like you understand; could you please explain what happened in your head?”

I am so excited to see what you can do and hear what you have to say.
Discussion Skills

1. Contribute to the class every day
2. Speak to classmates, not to the instructor
3. Put up a difficult problem, even if not correct
4. Use other students’ names
5. Ask questions
6. Answer other students’ questions
7. Suggest an alternate solution method
8. Draw a picture
9. Connect to a similar problem
10. Summarize the discussion of a problem
Real Analysis

Notation.
- A set is a notion that we won’t define, because any definition would end up using a word like “collection,” which we’d then need to define. We’ll just assume that we understand what is meant by a set, and let this notion of a set be fundamental.
- We use a capital letter to denote a set, e.g. “Let $S$ be the set of even numbers.”
- The symbol $\in$ means “is/be an element of,” and $\notin$ means “is not an element of.”
- We use a lower-case letter to denote an element of a set, e.g. “Let $a$ be an element of a set $A$.”
- To describe the elements of a set, use curly braces $\{\}$. We use a word like “collection,” which we’d then need to define. We’ll just assume that we understand what is meant by a set $S$, and let this notion of a set be fundamental.

1. Let $A = \{1, 2, 3, 4\}$. Which of the following are true statements?
   - (a) $\{4, 3, 2\} \in A$
   - (b) $5 \in A$
   - (c) $6 \in A$
   - (d) $7 \notin a$

Talking about sets.
- $X \subseteq Y$ is read “$X$ is a subset of $Y$,” and means that every $x$ in $X$ is also in $Y$.
- An equivalent notation to $X \subseteq Y$ is $X \subseteq Y$. If one wants to specify that $X \not\subseteq Y$, one can write $X \nsubseteq Y$. Otherwise, $X \subseteq Y$ allows for the possibility that $X = Y$.

2. (Continuation) Let $S$ be as above, and $C = \{1, 2, 3, 4, 5, 6\}$. Which are true? Explain.
   - (a) $3 \in B$
   - (b) $A \subset C$
   - (c) $C \subseteq B$
   - (d) $A \subset B$
   - (e) $B \subset A$

Useful sets.
- The empty set $\emptyset$, the set consisting of no elements.
- The natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. In Europe, $\mathbb{N}$ starts with 0.
- The integers $\mathbb{Z} = \{-3, -2, -1, -0, 1, 2, 3, \ldots\}$, from the German Zahl for number.
- The rationals $\mathbb{Q} = \{p/q \text{ in lowest terms} : p \in \mathbb{Z}, q \in \mathbb{N}\}$, from quotient
  $$= \{\text{repeating or terminating decimals}\}.$$
- The reals $\mathbb{R} = \{\text{all decimals}\}$, with the understanding that $0.999\ldots = 1,$ etc.

Note that these symbols are typeset as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and written by hand as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

4. (Continuation) Which of the following are true? Justify your answers.
   - (a) $B \subset \mathbb{N}$
   - (b) $C \subset \mathbb{Z}$
   - (c) $\mathbb{Q} \subset \mathbb{R}$
   - (d) $\mathbb{Z} \subset \mathbb{N}$
   - (e) $\emptyset \subset \mathbb{N}$
   - (f) $\emptyset \subset A$

Working with sets.
- The intersection $X \cap Y$ of two sets $X$ and $Y$ is the set of all elements that are in $X$ and in $Y$: $X \cap Y = \{x : x \in X \text{ and } x \in Y\}$.
- The union $X \cup Y$ of two sets $X$ and $Y$ is the set of all elements that are in $X$ or in $Y$: $X \cup Y = \{x : x \in X \text{ or } x \in Y\}$.
- The complement $X^C$ of a set $X$ is the set of points not in $X$: $X^C = \{x : x \notin X\}$. For this to make sense, the “universal set” that $X$ lives in must be understood.
- The set $X - Y$, or $X \setminus Y$, is the set of all points in $X$ that are not in $Y$. 

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5. Shade the regions corresponding to $X \cap Y$, $X \cup Y$, $X^C$, and $X - Y$, respectively.

6. Let $X$ be a subset of a universal set $U$, and let $X$ and $Y$ be subsets of $U$. Simplify:
   \begin{align*}
   (a) \quad & (X \cup Y) \cap (U - X) \\
   (b) \quad & X \cup (Y \cap X^C) \\
   (c) \quad & (X \cap Y) \cup (X \cap Y^C)
   \end{align*}

7*. Find infinitely many nonempty sets $S_1, S_2, \ldots$ of natural numbers such that
   \[ \mathbb{N} \supset S_1 \supset S_2 \supset S_3 \cdots \]
   and $\cap_{n=1}^{\infty} S_n = \emptyset$. Here the symbol $\cap_{n=1}^{\infty} S_n$ means $S_1 \cap S_2 \cap \cdots$, and is used to take the intersection of infinitely many sets.

Functions.
   \begin{itemize}
   \item A function from $X$ to $Y$ is a rule that assigns, to each $x \in X$, exactly one $y \in Y$.
   \item We write $f : X \to Y$, and if $f(x) = y$, we write $x \mapsto y$ which is read “$x$ maps to $y$.”
   \item If $f$ maps distinct points to distinct values, then $f$ is called one-to-one or injective. Equivalently, $f$ is injective if $f(x) = f(y)$ implies that $x = y$.
   \item $X$ is called the domain of $f$, and $Y$ is the codomain of $f$.
   \item The set of all outputs $f(X) = \{f(x) : x \in X\}$ is called the image of $f$. If the image is the entire codomain, $f$ is called onto or surjective. Equivalently, $f$ is surjective if, for each $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.
   \item A function that is both injective and surjective is called bijective.
   \end{itemize}

8. For each of the following, say whether it is injective, surjective, or both (bijective):
   \begin{align*}
   (a) \quad & f(x) = -x \\
   (b) \quad & f(x) = x^2 \\
   (c) \quad & f(x) = \sin x \\
   (d) \quad & f(x) = e^x \\
   (e) \quad & f(x) = x^3 + x^2.
   \end{align*}

Inverses.
   \begin{itemize}
   \item If $f : X \to Y$ is one-to-one and onto (bijective), then we can define the inverse of $f$ to be the function $f^{-1} : Y \to X$, defined such that $f^{-1}(y) = x$ when $f(x) = y$.
   \item We can define the image of a set $A \subset X$ as the collection of images of points in $A$, $f(A) = \{f(a) : a \in A\}$.
   \item No matter if $f$ is bijective or not, we can define the inverse image of a set $B \subset Y$ as the collection of points in $X$ that map to points in $B$: $f^{-1}(B) = \{x \in X : f(x) \in B\}$.
   \end{itemize}

9. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Find each of the following, or say why it is not possible:
   \begin{align*}
   (a) \quad & f([-3, -2]) \\
   (b) \quad & f^{-1}([0, 1]) \\
   (c) \quad & f^{-1}([-2, -1]) \\
   (d) \quad & f^{-1}(x) \text{ Hint: Draw a picture}
   \end{align*}
**Real Analysis**

**Infinite sets.** A set is called *countable* if it is finite, or if its elements can be put in one-to-one correspondence with the natural numbers. Equivalently, a set is countable if its elements can be *listed*, in a (possibly) infinite list. Otherwise, the set is called *uncountable*.

1. Show that the set of even natural numbers $S = \{2, 4, 6, \ldots\}$ is countable, by:
   (a) Showing how to systematically list them;
   (b) Explicitly constructing a bijective function from $\mathbb{N}$ to the even numbers.

2. (Continuation) Write a proof that the even numbers are countable, using your function from 1(b). The purpose of this problem is to practice constructing a clear, rigorous proof. Do this by filling in the following. Write the entire proof, not just the parts in the blanks.

   **Proof.** We will show that ____________________________________________________________________________,
   and showing that it ____________________________________________________________________________.

   Let $\mathbb{N}$ be the set of natural numbers, and let $S$ be the set of even numbers.
   Define $f : \mathbb{N} \rightarrow S$ by ____________________________________________________________________________.

   First, we will show that $f$ is injective. Suppose that $f(x) = f(y)$. Then ___________, so $x = y$, as desired.

   Now, we will show that $f$ is surjective. Let $x \in S$. Then ____________________________________________________________________________, so $x = f(n)$ for some $n \in \mathbb{N}$, as desired.

   Thus $f$ is injective and surjective, so $f$ is bijective, so there is a bijective function from $\mathbb{N}$ to the even numbers, so ____________________________________________________________________________, as desired.

3. This result seems to be a contradiction: the set of even numbers seems to be a smaller set than $\mathbb{N}$ (half as big!), and yet the two sets have the same size. Explain.

4. Show that the integers $\mathbb{Z}$ are countable.

5. Dramatic foreshadowing.
   (a) Show that the set $S_1 = \{1, 1/2, 1/3, 1/4, \ldots\} = \{1/n : n \in \mathbb{N}\}$ is countable.
   (b) Show that the set $S_2 = \{2, 2/2, 2/3, 2/4, \ldots\} = \{2/n : n \in \mathbb{N}\}$ is countable.
   (c) Show that the set $S_3 = \{3, 3/2, 3/3, 3/4, \ldots\} = \{3/n : n \in \mathbb{N}\}$ is countable.
   (d) Show that $S_1 \cup S_2 \cup S_3$ is countable, by showing how to put all the elements on a list.

**More dramatic foreshadowing: An algorithm.** Suppose that you are given a list of numbers. Maybe they are Dewey decimal numbers for books that are already in your library, and now you have a new book. So to make a number for the new book, you need to construct a number $x$ that you can be sure is not already on the list.

Here is an algorithm for constructing a number not on a given list:

- Make the $1$st digit of $x$ different from the $1$st digit of the $1$st number on the list.
- Make the $2$nd digit of $x$ different from the $2$nd digit of the $2$nd number on the list.
- Make the $3$rd digit of $x$ different from the $3$rd digit of the $3$rd number on the list, etc.
- For concreteness, let the $n$th digit of $x$ be 1, unless the $n$th digit of the $n$th number on the list happens to be 1, and in that case let the $n$th digit of $x$ be 2.
6. Use the algorithm to construct a number that is not on the example list below.

1. 3.1415926
2. 5.0000000
3. 2.7182818
4. 1.6180339
5. 1.2121212
6. 1.4142135
7. 0.6931471

7. Prove the following:

**Theorem.** Any subset of a countable set is countable.

*Proof.* We will show that any subset of a countable set is countable. *State the result.*

Let $X$ be the countable set, and let $E \subset X$ be the subset. *Define your terminology.*

First, suppose $E$ is finite. Then the result follows, because ______________.

Now, suppose $E$ is infinite. Then we can write $X = \{x_1, x_2, x_3, \ldots\}$, because ______________.

*Idea: The elements of $E$ are elements of $X$, so just re-list them.*

Now let $n_1$ be the smallest integer so that $x_{n_1} \in E$.

Let $n_2$ be the next-smallest integer so that $x_{n_2} \in E$, and so on.

Now we can list the elements of $E$: ______________, so $E$ is countable.

To explicitly construct a one-to-one correspondence between $\mathbb{N}$ and the elements of $E$, we can construct a function $f : \mathbb{N} \to E$ by $f(k) =$_______________. *Use definition of countable.*

$f(k)$ is one-to-one because ________________________________.

$f(k)$ is onto because ________________________________.

So $E$ is countable, as desired. *State result that you have proved.*

8. Prove that $0.999\ldots = 1$. The proof should start, “Let $x = 0.999\ldots$” and should end “Therefore $x = 1$, as desired.” You can find the proof on the Internet. Your job is to write it out carefully.

9. The ancient Greeks believed that all numbers were rational, and were horrified to discover that this is not the case, because $\sqrt{2}$ is not rational. Prove this fact by filling in the following in this *proof by contradiction*:

We will show that ________________.

Suppose, to yield a contradiction, that ________________.

Then we can write $\sqrt{2} = p/q$, where $p \in ____$, $q \in ____$ and $p/q$ is in lowest terms.

Squaring both sides yields ________________.

We can ________________ to obtain $2q^2 = p^2$, so $p$ is even.

Since $p$ is even, we can write $p = 2k$, for some $k \in ____$, yielding $2q^2 = (2k)^2$.

We can ________________ to obtain $q^2 = 2k^2$, so $q$ is even.

This contradicts our initial assumption that ________________.

Thus we triumphantly conclude that ________________, as desired.
1. Show that the set of positive rational numbers \( \mathbb{Q}^+ = \{ x : x \in \mathbb{Q} \text{ and } x > 0 \} \) is countable.

2. **Cantor’s diagonalization argument.** Prove the following theorem:

**Theorem.** The real numbers are uncountable.

**Proof.** We will show that the real numbers are uncountable. **State the result.**

We will use a proof by contradiction: we will suppose that the real numbers are countable, and obtain a contradiction. **Explain the plan.**

Suppose that the real numbers are countable. Then they can be listed, because __________ __________. **State definition of countability.**

→ Finish the proof, using the algorithm given for Page P1 # 6.

**Precise notions to bound sets.** Let \( A \) be a nonempty set of real numbers.

- A real number \( u \) is an **upper bound** for \( A \) if \( a \leq u \) for all \( a \in A \).
- A real number \( l \) is a **lower bound** for \( A \) if \( l \leq a \) for all \( a \in A \).
- A set is **bounded** if it has both an upper and a lower bound.
- A real number \( s \) is the **supremum** ("soo-pree-mum") or **least upper bound** of \( A \) if \( s \) is an upper bound for \( A \), and \( s \leq u \) for any other upper bound \( u \) of \( A \). The supremum is denoted \( \text{sup}(A) \) or \( \text{l.u.b.}(A) \).
- A real number \( t \) is the **infimum** ("in-fee-mum") or **greatest lower bound** of \( A \) if \( t \) is a lower bound for \( A \), and \( l \leq t \) for any other lower bound \( l \) of \( A \). The infimum is denoted \( \text{inf}(A) \) or \( \text{g.l.b.}(A) \).
- A real number \( m \) is the **maximum** of \( A \) if \( m \in A \) and \( a \leq m \) for all \( a \in A \).
- A real number \( n \) is the **minimum** of \( A \) if \( n \in A \) and \( n \leq a \) for all \( a \in A \).

Note that, if you are just trying to show that a set is bounded, a crazy big bound like 1000 works just as well as a bound like 1. There is no need to do extra work to find a tight bound.

3. Complete the following table by filling in each box with a number, the letters DNE for “does not exist,” or the word “Yes” or “No.” Be prepared to justify your answers.

<table>
<thead>
<tr>
<th>Set</th>
<th>L.B.</th>
<th>U.B.</th>
<th>min</th>
<th>max</th>
<th>sup</th>
<th>inf</th>
<th>is sup in set?</th>
<th>set bounded?</th>
</tr>
</thead>
<tbody>
<tr>
<td>{x \in \mathbb{R} : 0 \leq x &lt; 1}</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>{x \in \mathbb{R} : 0 \leq x \leq 1}</td>
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<tr>
<td>{x \in \mathbb{R} : 0 &lt; x &lt; 1}</td>
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<tr>
<td>{1/n : n \in \mathbb{Z} \setminus {0}}</td>
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<tr>
<td>{1/n : n \in \mathbb{N}}</td>
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<tr>
<td>{x \in \mathbb{R} : x &lt; \sqrt{2}}</td>
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<tr>
<td>{1, 4, 6, 16, 25}</td>
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<tr>
<td>{(-1)^n(2 - 1/n) : n \in \mathbb{N}}</td>
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<tr>
<td>{\ln(x) : x \in \mathbb{R}, x &gt; 0}</td>
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<td></td>
</tr>
<tr>
<td>{e^x : x \in \mathbb{R}}</td>
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</tbody>
</table>
4. For each of the following statements, either say it is true and explain why, or say it is false and provide a counterexample. Hint: Consider the examples you explored in the table.

(a) Every set has a maximum.
(b) Every set has a minimum.
(c) If a set is bounded, then it has a supremum.
(d) If a set is bounded, then it has an infimum.
(e) If a set has an infimum, then it is bounded below.
(f) If a set has a supremum, then it is bounded above.
(g) If a set is bounded, then it has both a maximum and a minimum.
(h) If a set has a maximum, then it is bounded above.
(i) If a set is bounded above, then it has a maximum.

Measuring distance. A metric space is a set $E$, along with a rule that assigns, to each pair $p, q \in E$, a real number $d(p, q)$, called the distance function $d : E \times E \to \mathbb{R}$, such that:

1. $d(p, q) \geq 0$ for all $p, q \in E$
2. $d(p, q) = 0$ if and only if $p = q$ 
3. $d(p, q) = d(q, p)$
4. $d(p, r) + d(r, q) \geq d(p, q)$ for all $p, q, r \in E$.

5. Check the four requirements above, for each of the following metrics on $\mathbb{R}^2$:

(a) The standard Euclidean metric (i.e. using the Pythagorean theorem),
(b) The “Taxicab metric”: $d((a, b), (c, d)) = |c - a| + |d - b|$ (also explain the name).

6. Check the four requirements for the following:

(a) In $\mathbb{R}^n$, $d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sup\{|y_1 - x_1|, \ldots, |y_n - x_n|\}$.
(b) For any set $E$, and any $x, y \in E$, $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$

Topology: the notion of open sets. Given a metric space $E$, a point $p_0 \in E$, and a real number $r > 0$, the open ball in $E$ of center $p_0$ and radius $r$ is $B_r(p_0) = \{p \in E : d(p_0, p) < r\}$.

7. Sketch the following open balls.

(a) In $\mathbb{R}$, the set $B_1(0)$.
(b) In $\mathbb{R}^2$, the set $B_{1/2}(1, 1)$.

8. Express the open interval $(1, 2) \in \mathbb{R}$ as an open ball as above. Then do the same for the general open interval $(a, b)$. 

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1. Show that $Q$ is countable. *Hint:* Follow the idea of Page P1 # 5.

(Okay, sure, the set of even numbers is the same size as the set of natural numbers, even though it seems half as big. But $Q$ seems *much* bigger than $N$! This result is astonishing.)

2. A new metric space.

Let $\mathcal{B} = \{\text{bounded, real-valued functions on } \mathbb{R}\}$

$$= \{f : \mathbb{R} \to \mathbb{R} \text{ such that there exists } M \in \mathbb{R} \text{ with } |f(x)| < M \text{ for all } x \in \mathbb{R}\}.$$ 

Define $d(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}$. Show that $d$ is a metric on $\mathcal{B}$. *Hint:* proving parts (1)-(3) from the definition are straightforward, and part (4) requires some work.

**The mathematical “or.”** In mathematics, “or” means one, or the other, or both. Shall we meet to do Real Analysis on Monday or Tuesday? Both!

**Implication.** There are many ways to say that one statement $A$ implies another statement $B$. The following all mean exactly the same thing:

- If $A$, then $B$.
- $A$ implies $B$, written $A \implies B$.
- $A$ only if $B$.
- $B$ if $A$, written $B \iff A$.
- not $B$ implies not $A$. (This is the *contrapositive*)

3. Let statement $A$ be “the number $n$ is divisible by four,” and let statement $B$ be “the number $n$ is even.” Write out the five implications above, using these statements. Considering this example, do you agree that they are all logically equivalent?

**True and false.** An implication is true if $B$ is true, or if $A$ is false (in which case we say that the implication is “vacuously true.”) For example, the statement “If 5 is even, then 15 is prime” is vacuously true. An implication is false only if $A$ is true and $B$ is false.

4. Is the statement:

If $x \in \mathbb{Q}$, then $x^2 \in \mathbb{N}$

true or false for the following values of $x$? Justify your answers.

(a) $x = 1/2$ \hspace{1cm} (b) $x = 2$ \hspace{1cm} (c) $x = \sqrt{2}$ \hspace{1cm} (d) $x = \sqrt[3]{2}$

**Open sets.** A subset $S$ of a metric space $E$ is *open* if, for all $p \in S$, $S$ contains some open ball with center $p$. In other words, for all $p \in S$, there exists an $r > 0$ such that $B_r(p) \subset S$.

5. Prove that the empty set is open. *Hint:* vacuously true

6. Let $S \subset \mathbb{R}$ be defined by $S = [0, \infty)$. Show that $S$ is *not* an open set. *Hint:* find a point in $S$ about which there is no open ball that is completely contained in $S$.

7. Let $E = [0, \infty)$. Prove that, in $E$, $E$ is open.

8. Prove that, for any metric space $E$, the entire space $E$ is an open set.
9. In each of the following pairs, exactly one of the statements is true. For the one that is true, explain why; for the one that is false, provide a counterexample.

1. If a set of real numbers is bounded above, then it has a maximum.
2. If a set of real numbers is bounded above, then it has a supremum.

1. If a set of real numbers has a supremum, then it has a maximum.
2. If a set of real numbers has a maximum, then it has a supremum.

1. If a set of real numbers has an infimum, then the infimum is in the set.
2. If a set of real numbers has a minimum, then the minimum is in the set.

10. Vacuously true statements can be used to make hilarious jokes, with other people who also understand vacuously true statements. Explain each of the following jokes:

- All of my black cars are Maseratis.
- I got an A in every underwater basket weaving course I took.
- Vassar Football, undefeated since 1861.
1. Prove that any open ball \( B_R(p_0) \) in any metric space \( E \) is an open set. *Hint:* for any point \( p \) in the ball, find its distance to the boundary of the ball using \( p_0 \) and \( R \), and choose a smaller number as the radius of the open ball around \( p \).

**Definition.** The Cartesian product \( X \times Y \) of two sets \( X \) and \( Y \) is the set
\[
X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.
\]

2. Show that the Cartesian product of two countable sets is countable.

3. **Theorem.** The union of any collection of open sets is open.

In order to even start talking about this, we need a way of indexing an arbitrary collection of sets. We can’t use \( \bigcup_{n=1}^{k} \) or \( \bigcup_{n=1}^{\infty} \), because the collection might be uncountable. The way we get around this is to use the symbol \( \alpha \) to index the collection: \( \bigcup_{\alpha} A_{\alpha} \).

**Proof.** We will show that

We will do this by showing that, for any \( p \) in the union, there is an open ball around \( p \) contained in the union. Let \( p \in \bigcup_{\alpha} A_{\alpha} \). Then for some \( \alpha_0 \), \( p \in A_{\alpha_0} \), because \( \bigcup_{\alpha} A_{\alpha} \).

Since \( A_{\alpha_0} \) is open, . . . (complete the proof).

**Converse and logical equivalence.** The converse of the statement “\( A \) implies \( B \)” is the statement “\( B \) implies \( A \).” If a statement and its converse are both true, we say \( A \) and \( B \) are *logically equivalent*, or in other words \( A \iff B \), or in other words “\( A \) if and only if \( B \).”

4. Let statement \( A \) be “\( x \) is a Swarthmore student” and let statement \( B \) be “\( x \) is a human being.” Write out the implication \( A \implies B \), its contrapositive, and its converse. Which of these implications are true?

5. For which real values of \( x \) is the converse of the statement of Page 3 \# 4 true?

6. Let \( S \subset \mathbb{Z} \). Consider the statements:

   \( A \): All elements of \( S \) are even. (For all \( x \in S \), \( x \) is even.)

   \( B \): Some element of \( S \) is even. (There exists an \( x \in S \) such that \( x \) is even.)

   (a) Does \( B \implies A \)?    (b) Does \( A \implies B \)?

7. Suppose that the algorithm for Cantor’s diagonalization argument had, instead of specifying 1s and 2s, just said “make sure the \( n \)th digit of your number \( x \) is different from the \( n \)th digit of the \( n \)th number on the list.” Suppose that your plan is to make the \( n \)th digit of \( x \) a 9, unless the corresponding digit is a 9, in which case you will make the \( n \)th digit of \( x \) a 2. Explain how the algorithm could “accidentally” construct a number already on the list. *Hint:* Put the number 1.0000... on your list, choose 0 as the integer part of your number, and refer to Page P1 \# 8.
1. **Theorem.** The intersection of a *finite* number of open sets is open.

*Proof.* We will show that (a) ______________. We will do this by showing that, for any \( p \) in the intersection, there is an open ball around \( p \) contained in the intersection. Let \( p \in \bigcap_{k=1}^{n} A_k \).

Then \( p \in A_k \) for all \( 1 \leq k \leq n \), because (b) ______________. Thus, there exist \( r_k \in \mathbb{R} \) such that \( B_{r_k}(p) \subset A_k \) for each \( k \). (c) (complete the proof) *Hint:* Draw a picture.

2. Consider the (false!) statement: *The intersection of infinitely many open sets is open.*

(a) Explain where the proof above breaks down for infinitely many sets.

(b) Give a counterexample to the statement.

**Closed sets.** A subset \( S \) of a metric space \( E \) is *closed* if \( S^C \) is open. A *closed ball* in a metric space \( E \), with center \( p_0 \) and radius \( r \), is the set of points \( \{ p \in E : d(p,p_0) \leq r \} \).

3. Prove that the empty set is closed.

4. Prove that, for any metric space \( E \), the entire space \( E \) is a closed set.

5. We have now proved that the empty set is both open and closed, and also any entire space \( E \) is both open and closed. Are these contradictions? Explain.

6. Prove that any closed ball \( S = B_R(p_0) \) is a closed set. *Hint:* for any \( p \) in \( B_R(p_0)^C \), find the distance to \( p_0 \), subtract this distance from \( R \), and draw a picture as in Page 4 # 1.
Completeness axiom. A nonempty set of real numbers that is bounded from above has a least upper bound.

7. **Theorem.** Let $S$ be a nonempty, closed subset of $\mathbb{R}$ that is bounded from above. Then $S$ has a maximum element.

**Proof.** We will show that any nonempty, closed subset $S \subset \mathbb{R}$ that is bounded from above has a maximum element. We will do this by showing that the least upper bound $a$ of $S$ is contained in $S$. The proof will be by contradiction: we will first suppose that $a \notin S$, and derive a contradiction. (Fill in the reasoning steps in the following proof.)

Let $a$ be the least upper bound of $S$. We know that the least upper bound exists, by the completeness axiom. We want to show that $a \in S$. Suppose, for a contradiction, that $a \notin S$. $S^C$ is open, because (a) __________. Thus there exists some $r > 0$, such that $B_r(a) \subset S^C$, because (b) __________. But then $a - r$ is also an upper bound for $S$, because (c) __________. This contradicts $a$ being the least upper bound for $S$, because (d) __________. Thus $a \in S$, and thus $S$ has a maximum element, as desired.

8. (Continuation) The statement of the theorem contains the conditions that $S$ is a nonempty, closed subset of $\mathbb{R}$. Give a counterexample or explanation for why the statement “$S$ has a maximum element” fails to be true if we remove the assumption that:

(a) $S$ is nonempty;  (b) $S$ is closed;  (c) $S$ is a subset of $\mathbb{R}$.

9. Find the error in the following “proof” that the rational numbers are uncountable: Suppose the rational numbers are countable. Then we can list them. Use the same construction as in Page 2 # 2 to find a number not on the list. Thus the rational numbers are uncountable.
1. Prove that the union of a finite number of closed sets is closed. *Hint:* First, argue that \( \bigcup_{k=1}^{n} A_k^c \) = \( \bigcap_{k=1}^{n} A_k^c \). Then argue that, if each \( A_i \) is closed, this intersection is open.

2. Prove that the intersection of any collection of closed sets is closed.

3. For a point \( p \in \mathbb{R}^n \), consider the set \( S = \bigcap_{m=1}^{\infty} \{x \in \mathbb{R}^n : d(x, p) \leq 1/m\} \).
   (a) Prove that \( S \) is closed. *Hint:* Quote a recent result you have proved.
   (b) Give a simple description of the set \( S \).
   (c) Prove that any single point \( \{p\} \), where \( p \in \mathbb{R}^n \), is a closed set.

**Bounded sets in general.** A subset \( S \) of a metric space \( E \) is *bounded* if it is contained in some ball, or in other words if there exists \( p \in E \) and \( r > 0 \) such that \( S \subset B_r(p) \).

4. Show that the set \( \{1000 - 500/n^2 : n \in \mathbb{N}\} \) is bounded by finding a suitable \( p \) and \( r \).

**Limits and convergence.** So far, the only way we can tell that a set is closed is by looking at its complement. Soon, we will learn a new way, using sequences and limits. Let \( p_1, p_2, \ldots \) be a sequence of points in a metric space \( E \).

- A point \( p \in E \) is a *limit* of \( \{p_i\}_{i=1}^{\infty} \) if, for any \( \epsilon > 0 \), there exists \( N > 0 \) such that, whenever \( i > N \), \( d(p, p_i) < \epsilon \).
- In such a situation, we say that the sequence \( p_i \) *converges to* \( p \), and write \( \lim_{i \to \infty} p_i = p \).

5. I like to think of sequence convergence as a competition between me and an interlocutor:
   ME: Consider sequence \( \{p_k\}_{k=1}^{\infty} \), where \( p_k = (-1)^k/k \). I claim this sequence converges to 0.
   INTERLOCUTOR: Nonsense! Show me that the terms get within 0.1 of 0.
   ME: Okay, take \( k > 10 \). After that, the terms are all closer than 0.1.
   INTERLOCUTOR: Hmm! Now show me that the terms get within 0.0001 of 0.
   ME: Okay, then take \( k > 10000 \). After that, the terms are all closer than 0.0001.
   INTERLOCUTOR: Hmm. Show me that the terms get within \( \epsilon \) of 0, for any \( \epsilon > 0 \) that I might ever suggest.
   ME: Okay, take \( k > \) ___________, and after that the terms are all closer than \( \epsilon \).

6. Prove, from the definition (i.e. by finding an \( N \) that depends on the given \( \epsilon \), sometimes called \( N(\epsilon) \) to emphasize this dependence), that the sequence \( p_n = 1000/n^3 \) converges to 0.

7. Which of the following statements are true of the real numbers?
   (a) For all \( x \), exists \( y \) such that \( y > x^2 \).  
   (b) There exists \( y \) such that, for all \( x \), \( y > x^2 \).
   (c) There exists \( y \) such that, for all \( x \), \( y < x^2 \).
1. Consider the (false!) statement: *The union of infinitely many closed sets is closed.*

(a) We proved in Page 5 # 1 that the union of *finitely* many closed sets is closed. Explain where the proof breaks down for infinitely many sets.

(b) Give a counterexample to the statement.

2. Prove that any finite set of points \( \{p_1, \ldots, p_k\} \subset \mathbb{R}^n \) is closed.

3. For each of the following sequences, say whether it converges or diverges. For those that converge, prove that it converges by finding the limit \( p \) and finding an \( N(\epsilon) \) for any \( \epsilon > 0 \).

(a) \( a_n = \frac{\sin n}{n} \)

(b) \( b_n = 1 + (-1)^n \)

(c) \( 1, 0, 1/2, 0, 1/4, 0, 1/8, 0, \ldots \)

4. **Theorem.** A sequence \( \{p_i\}_{i=1}^{\infty} \) of points in a metric space \( E \) has at most one limit.

**Proof.** We will show that a sequence of points in a metric space \( E \) has at most one limit. We will do this by contradiction, by supposing that it has two different limits, and showing that the two limits must be the same, by showing that the distance between them is 0.

Suppose that the sequence \( \{p_i\}_{i=1}^{\infty} \) in metric space \( E \) has two different limits, \( p \) and \( p' \). By definition of \( p \) and \( p' \) each being a limit point, we know that:

Given any \( \epsilon/2 > 0 \), there exists \( N \) such that \( d(p, p_i) < \epsilon/2 \) for all \( i > N \), and given any \( \epsilon/2 > 0 \), there exists \( M \) such that \( d(p', p_i) < \epsilon/2 \) for all \( i > M \).

Given \( \epsilon > 0 \), choose a number \( n > \max\{N, M\} \). (Finish the proof. Note that we used \( \epsilon/2 \) to find \( N \) and \( M \) so that the bound will come out cleanly to \( \epsilon \) at the end, but it would also have been fine to use \( \epsilon \) and come out with a bound of \( 2 \epsilon \) at the end.)

**Bounded sequences.** A sequence of points \( \{p_i\}_{i=1}^{\infty} \) in a metric space is *bounded* if it is bounded as a set, i.e. if it is contained in a ball.

5. **Theorem.** Every convergent sequence is bounded.

**Proof.** We will show that every convergent sequence is bounded. We will do this by constructing a ball that contains all of the points of the sequence. Take \( \epsilon = 1 \). Then there exists \( N \) such that, for all \( n > N \), \( d(p_n, p) < 1 \), because (a) \( r = \max\{1, d(p, p_1), d(p, p_2), \ldots, d(p, p_n)\} \). Now take \( r = \max\{1, d(p, p_1), d(p, p_2), \ldots, d(p, p_n)\} \). (b) (finish the proof)

6. (Continuation) State the converse (note: converse, not contrapositive) of the theorem in Problem 5. Then either prove it or give a counterexample.

7. Consider a metric space \( E \) with the metric: \( d(p, q) = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p \neq q. \end{cases} \)

(a) What do the open balls look like in this space?

(b) What do the closed balls look like in this space?

(c) True or False: Any finite set of points, in a metric space \( E \) with this metric, is open.

8. Is all of \( \mathbb{R} \) the only open set that contains all of \( \mathbb{Q} \)? Prove your answer correct.
1. Give an example of each of the following:
   (a) Sets \( A \subset B \) with \( \text{sup}(A) < \text{sup}(B) \);   
   (b) Sets \( A \subsetneq B \) for which \( \text{sup}(A) = \text{sup}(B) \).

2. Prove or give a counterexample: If \( \lim_{n \to \infty} (a_{n+1} - a_n) = 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

3. **Theorem.** If \( a_n \in \mathbb{R} \) and \( c \in \mathbb{R} \), and \( \lim_{n \to \infty} a_n = a \), then \( \lim_{n \to \infty} c \cdot a_n = c \cdot a \).

   **Scratchwork:** We want to show that, given \( \epsilon > 0 \), we can find an \( N \) large enough that, for all \( n > N \), \( |c \cdot a_n - c \cdot a| < \epsilon \), which will hold if \( |a_n - a| < \epsilon/|c| \). Okay, ready for the proof.

   **Proof.** We will show that, if the real sequence \( a_n \to a \), then \( c \cdot a_n \to c \cdot a \), for any \( c \in \mathbb{R} \). We may assume that \( c \neq 0 \), because when \( c = 0 \), \( (a) \) . Given any \( \epsilon > 0 \), choose \( N \) so that \( |a_n - a| < \epsilon/|c| \) for all \( n > N \). We can do this because \( (b) \) . Then, for any \( n > N \), we have \( |c \cdot a_n - c \cdot a| = (c) < \epsilon \), as desired.

4. **Subsequences.** We like sequences to converge, but most don’t. Fortunately, most sequences have subsequences that do converge. Given a sequence \( a_n \), a *subsequence* \( a_{m_n} \) consists of some (infinitely many) of the terms, in the same order.

   **The lim sup and lim inf.** Even for sequences that are not convergent, sometimes elements of the sequence do accumulate. The following *always* exist:

   - The **lim inf** of a sequence is the smallest limit of any subsequence, or \( \pm \infty \).
   - The **lim sup** of a sequence is the largest limit of any subsequence, or \( \pm \infty \).

4. In the following table, write out the first 8 terms of any sequence for which they are not already written out for you. Then find the lim inf and lim sup of each sequence.

<table>
<thead>
<tr>
<th>sequence</th>
<th>first 8 terms</th>
<th>lim inf</th>
<th>lim sup</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n = 1/n )</td>
<td>( 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots )</td>
<td>( 1 )</td>
<td>( 1/8 )</td>
</tr>
<tr>
<td>( a_n = \sin(n \pi/2) )</td>
<td>( 1, -1, 2, -2, 3, -3, \ldots )</td>
<td>( -1 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( a_n = -n^2 )</td>
<td>( 1, -1, 2, -2, 3, -3, \ldots )</td>
<td>( 1 )</td>
<td>( -1 )</td>
</tr>
<tr>
<td>( a_n = 2 + \frac{(-1)^n}{n} )</td>
<td>( 1, \frac{3}{2}, \frac{3}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, \frac{1}{2}, \ldots )</td>
<td>( 2 )</td>
<td>( 3 )</td>
</tr>
<tr>
<td>( a_n = \begin{cases} 3 - e^{-n} &amp; \text{if } n \text{ is even} \ 3 &amp; \text{if } n \text{ is odd} \end{cases} )</td>
<td>( 1, \frac{3}{2}, \frac{3}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, \frac{1}{2}, \ldots )</td>
<td>( 1 )</td>
<td>( 3 )</td>
</tr>
</tbody>
</table>

5. Make a conjecture, based on these examples, as to what conditions ensure that the lim inf of a sequence equals its lim sup.

**Monotonicity.** We use the following terms to describe sequences:

- A sequence \( \{a_i\}_{i=1}^\infty \) is **increasing** if \( a_1 \leq a_2 \leq a_3 \leq \ldots \)
- A sequence \( \{a_i\}_{i=1}^\infty \) is **decreasing** if \( a_1 \geq a_2 \geq a_3 \geq \ldots \)
- A sequence is **monotone** if it is either increasing or decreasing.

6. Prove or give a counterexample: Every convergent sequence of real numbers is monotone.
1. Prove that for all sequences \( \{a_n\} \subset \mathbb{R} \), \( \lim \inf \{a_n\} \leq \lim \sup \{a_n\} \).

2. State and prove your conjecture from Page 7 # 6.

**Sequence definition of a closed set.** Previously, if we wanted to know if a set was closed, the only way was to check if its complement was open. Now we will have another way. In fact, the following Theorem is sometimes used as the definition of a closed set.

3. **Theorem.** Let \( S \) be a subset of a metric space \( E \). Then \( S \) is closed if and only if every convergent sequence of points from \( S \) converges to a point in \( S \).

**Proof.** This is an “if and only if” statement, so we need to show both directions of implication.

(\(\implies\)) Suppose \( S \) is closed. We will show that every convergent sequence converges to a point in \( S \). We will prove this by contradiction: We will assume that some sequence of points in \( S \) converges to a point not in \( S \), and show that some of the points in the sequence (in fact, the entire “tail of the sequence”) are not in \( S \), which will be the desired contradiction.

Let \( \{p_n\} \subset S \) be a convergent sequence of points from \( S \). Let \( \lim_{n \to \infty} p_n = p \), and assume that \( p \notin S \). Thus, \( p \in S^C \). \( S^C \) is open, because \( a \), so there exists \( \epsilon > 0 \) such that \( B_\epsilon(p) \subset S^C \), because \( \epsilon \). But by the definition of a limit, there exists \( N \) such that for all \( n > N \), \( d(p_n, p) < \epsilon \). (finish the proof)

(\(\impliedby\)) Suppose that every convergent sequence of points in \( S \) converges to a point in \( S \). We will show that \( S \) is closed. We will prove the contrapositive: We will assume that \( S \) is not closed, and find an convergent sequence of points in \( S \) that converges to a point not in \( S \).

Suppose that \( S \) is not closed. Then \( S^C \) is not open. Thus, there exists a point \( p \in S^C \) such that every open ball around \( p \) contains a point from \( S \), because \( \epsilon \). So we can construct an infinite set of balls, \( \{B_{1/n}(p)\} \), and an infinite sequence of points \( p_n \), such that \( p_n \in B_{1/n}(p) \) for each \( n \in \mathbb{N} \), because \( \epsilon \). Then \( \{p_n\} \subset S \), but \( p_n \to p \) and \( p \notin S \). (finish the proof).

4. Prove that if \( \{a_i\} \) and \( \{b_i\} \) are convergent sequences of real numbers with limits \( a \) and \( b \) respectively, and if \( a_i \leq b_i \) for all \( i \), then \( a \leq b \).

5. Prove or give a counterexample for the statement above, with “\(\leq\)” replaced by “\(<\).”

6. **Theorem.** A bounded, monotone sequence of real numbers is convergent.

We will prove this for an increasing sequence; the proof for a decreasing sequence is similar. **Proof.** Idea of proof: Let \( a = \sup\{a_k\} \). We will show that \( \lim_{k \to \infty} a_k = a \). Given any \( \epsilon > 0 \), we need to show that there exists \( N \) such that, for any \( n > N \), \( |a - a_n| < \epsilon \). Now \( a - \epsilon \) is too small to be a lower bound for \( \{a_k\} \), because \( \epsilon \). So there exists \( N \) such that \( a_N > a - \epsilon \), because \( \epsilon \). So we know that \( a_i > a - \epsilon \) for all \( i > N \), because \( \epsilon \). Thus \( a - \epsilon < a_i < a + \epsilon \) for all \( i > N \). (finish the proof)

7. Consider \( \mathbb{R} \) with the metric \( d(p, q) = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p \neq q. \end{cases} \). Does the sequence \( a_n = 1/n \) have a limit in this metric space? (In other words, does the previous Theorem apply?)
1. Prove that if $a_n \leq b_n \leq c_n$ for each $n \in \mathbb{N}$, and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$, then $\lim_{n \to \infty} b_n = L$.

2. **Proposition.** (We use “Proposition” for results that are smaller than a Theorem.) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

We will first show that every bounded sequence of nonnegative real numbers in $\mathbb{R}$ has a convergent subsequence. We will do this by explicitly constructing a convergent subsequence. Consider a nonnegative sequence $a_1, a_2, a_3, \ldots$. Each $a_n$ starts off with a nonnegative integer before the decimal point, followed by infinitely many digits (possibly 0) after the decimal point. Since the sequence $a_n$ is bounded, some integer part $D$ before the decimal place occurs infinitely many times, because (a) . Throw away the rest of the $a_n$. Among the infinitely many remaining $a_n$ that start with $D$, some first decimal place $d_1$ occurs infinitely many times. Throw away the rest of the $a_n$.

(b) Complete the construction of a number $L = D.d_1d_2d_3\ldots$, and prove that there is a subsequence of $a_n$ converging to $L$.

(c) Now show that every bounded sequence of real numbers has a convergent subsequence, to complete the proof of the Proposition as stated.

**Accumulation points.** Let $S$ be a subset of a metric space $E$. A point $p \in E$ is an accumulation point of $S$ if, for every $\epsilon > 0$, $B_\epsilon(p)$ contains an infinite number of points from $S$. (An accumulation point is also called a cluster point or limit point.)

3. For each of the following sets, describe its set of accumulation points.
   (a) $\mathbb{Q}$ (b) the irrationals (c) $(a, b)$ (d) $\{1\}$ (e) $\{1/n : n \in \mathbb{N}\}$ (f) $\mathbb{Z}$

**The boundary, interior and closure.** Let $S$ be a subset of a metric space $E$. A point $p \in E$ is a boundary point of $S$ if every open ball about $p$ contains points of $S$ and points of $S^c$. The boundary of $S$, denoted $\partial S$, is the collection of all of the boundary points of $S$. The closure of $S$, denoted $\overline{S}$, is $S \cup \partial S$. The interior of $S$, denoted $\overset{\circ}{S}$, is $S \setminus \partial S$.

4. Find $\partial S, \overline{S}$ and $\overset{\circ}{S}$ for each set $S$: (a) $(0, 1]$ (b) $\mathbb{Z}$ (c) $\mathbb{Q}$ (d) $B_1(0, 0)$ in $\mathbb{R}^2$

5. Prove that every point in a set is either a boundary point or an interior point.

6. Prove that the interior of $S$ is the largest open set contained in $S$. *Hint:* first prove that the interior is open.

**Isolated points.** A point $p \in S$ is isolated if there exists $r > 0$ such that $p$ is the only point of $S$ in $B_r(p)$.

7. Prove that every boundary point is either an isolated point or an accumulation point.

8. Prove that “the limit of a sum is the sum of the limits”: If $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers, with $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$, then $\lim_{n \to \infty} (a_n + b_n) = a + b$. *Hint:* At the end, you will want to show that $|(a_n + b_n) - (a + b)| < \epsilon$, so break it into two parts using rules of absolute values, and take $n$ large enough that certain quantities are less than $\epsilon/2$.

9. (Continuation) Explain why, in the previous problem, $\{a_n\}$ and $\{b_n\}$ had to be sequences of real numbers, rather than just sequences in an arbitrary metric space.
Real Analysis

Cauchy sequences. A sequence \( \{p_n\}_{n=1}^{\infty} \) in a metric space \( E \) is Cauchy (pronounced “CO-shee”) if, for any \( \epsilon > 0 \), there exists \( N \) such that, if \( m,n > N \), then \( d(p_m,p_n) < \epsilon \). You can think of a Cauchy sequence as one that is “trying” to converge, but to a limit that is outside of its metric space.

1. Consider the sequence \( a_n = 1/n \) in \( \mathbb{R}^+ \) with the usual distance metric.
   (a) Show that \( \{a_n\} \) is a Cauchy sequence.
   (b) Show that \( \{a_n\} \) does not converge in this metric space.

2. Prove that a convergent sequence in any metric space is Cauchy.

3. Prove that a Cauchy sequence in any metric space is bounded. Hint: Use the same proof as we did to show that a convergent sequence is bounded.

4. Prove that a Cauchy sequence that has a convergent subsequence is itself convergent.

Set equivalence. To prove that two sets \( A \) and \( B \) are equal, you must prove set inclusion in both directions: \( A \subset B \) and \( B \subset A \).

5. Prove that \( (A \cup B)^C = A^C \cap B^C \). The proof must have two parts: First, suppose that \( x \in (A \cup B)^C \) and show that \( x \) must also be in \( A^C \cap B^C \). Then, suppose that \( x \in A^C \cap B^C \) and show that \( x \) must also be in \( (A \cup B)^C \).

6. Prove that \( \overline{S} = \bigcap_{C \text{ closed}, S \subset C} C \). First write out the statement in words. Then prove it.

Sometimes this is used as the definition of the closure of \( S \).

Covers. A collection of sets \( \mathcal{G} = \{G_\alpha\} \) is a cover of \( S \) if \( S \subset \bigcup_\alpha G_\alpha \). \( \mathcal{G} \) is an open cover if all of the \( G_\alpha \) are open sets.

7. Construct an open cover that:
   (a) covers \( \mathbb{R} \) with unit intervals
   (b) covers \( \mathbb{R}^2 \) with unit balls
   (c) covers \( \{1/n : n \in \mathbb{N}\} \)

Compactness. A subset \( X \) of a metric space \( E \) is compact if every open cover of \( X \) has a finite subcover. It turns out that the notion of compactness is extremely useful in analysis. Later, we will show that several other criteria are equivalent to compactness. The definition here is nice because it requires only the notion of an open set. However, this definition is rather difficult to check. Sure, we can find an open cover, but how do you check that every possible open cover has a finite subcover?

8. Show that \( (0,1] \) is not compact, by showing that the open cover \( \bigcup_{n=1}^{\infty} U_n \), where \( U_n = \{(1/n, \infty)\} \), has no finite subcover, i.e. that no finite subset of the \( U_n \)’s covers the set.

9. Show that \( \mathbb{R} \) is not compact, by finding an open cover that has no finite subcover.
1. **Proposition.** An infinite subset of a compact metric space has at least one accumulation point.

   **Proof.** We will prove this by contradiction, by assuming that the subset has no accumulation point, and showing that the subset must be finite. Suppose that $A \subset E$ is an infinite set with no accumulation point. Then for each $p \in E$, there exists an $r_p$ such that $B_{r_p}(p)$ contains only finitely many points of $A$, because (a) ___________. Then $\bigcup_{p \in E} B_{r_p}(p)$ is an open cover of $E$, because (b) ___________. Since $E$ is compact, there is a finite subcover $B_{r_1}(p_1) \cup \cdots \cup B_{r_k}(p_k)$ that covers $E$. (c) (finish the proof).

2. **Complete metric spaces.** A metric space $E$ is *complete* if every Cauchy sequence of points in $E$ converges to a point in $E$.

3. **Theorem.** $\mathbb{R}$ is complete. (Prove this) **Hint:** One method is to use the results of Page 9 # 3 and Page 8 # 2 and argue (using an $\epsilon - N$ proof) that every sequence converges to the same limit as its convergent subsequence.

4. **Corollary.** $\mathbb{R}^n$ is complete. (We use “Corollary” for a result that follows directly from an existing Proposition or Theorem.) Prove this. **Hint:** Do as in the Theorem, for each coordinate.

5. **Proposition.** $S = \{0\} \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$ is a compact set.

   **Proof.** We will show that $S$ is compact, by explicitly constructing a finite subcover from any open cover. Consider an arbitrary open cover $\bigcup G_\alpha$ of $S$. 0 must be in some open set $G_\alpha$, because (a) ___________, so call this open set $G_{\alpha_0}$. Then there exists an open ball $B_r(0) \subset G_{\alpha_0}$, because (b) ___________. Then for all $n > 1/r$, we have $1/n \in B_r(0)$, because (c) ___________. So $\{0\} \cup \{1/n : n > 1/r\} \subset G_{\alpha_0}$. (d) (finish the proof).

6. **Proposition.** $(0,1)$ is not compact, by finding an open cover with no finite subcover.
1. Write the contrapositive of each of the following implications. *Hint:* first write them in the form $A \implies B$, then write the contrapositive, then translate back into everyday English.

(a) If it ain’t broke, don’t fix it.
(b) No good deed goes unpunished.
(c) The early bird gets the worm.
(d) People who live in glass houses shouldn’t throw stones.

2. Write the negation of each statement. The way I like to think about negation is that someone says the statement $X$, and then you say, “no, you’re wrong, [negation of $X$].”

(a) Everyone in the class is named John.
(b) Someone in the class is named Jane.
(c) Every open cover has a finite subcover.
(d) There exists an open ball about $p$ such that $p$ is the only point of $S$ in the ball.

The following three results follow from the Proposition in Page 10 # 6.

3. **Corollary.** Every sequence of points in a compact metric space has a convergent subsequence.

   *Proof.* We will prove the result by explicitly constructing the convergent subsequence. Let $p_n$ be a sequence in a compact metric space $E$. There are two cases, depending on whether the number of different points in the sequence is finite or infinite. If $\{p_1, p_2, p_3, \ldots\}$ is a finite set, then there must be some point $p$ that occurs infinitely many times. (finish the proof.)

   On the other hand, if $\{p_1, p_2, p_3, \ldots\}$ is an infinite set, then it must have at least one accumulation point $p$, because (a) __________. Choose $n_1$ so that $p_{n_1} \in B_1(p)$. Choose $n_2 > n_1$ so that $p_{n_2} \in B_{1/2}(p)$, and so on. Note that for each $k$, there are infinitely many points of the sequence in $B_{1/k}(p)$, because (b) __________. (c) (finish the proof).

4. **Corollary.** A compact metric space is complete.

   *Proof.* We will show that a compact metric space is complete, by showing that every Cauchy sequence converges to a point in the space. *Hint:* combine two previous results.

5. **Corollary.** A compact subset of a metric space is closed.

   *Proof.* We will prove this by showing that every convergent sequence of points from a compact subset converges to a point in the compact subset (using the sequence definition of a closed set; see Page P3 # 3). *Hint:* apply a previous result.

6. Prove that a compact subset of a metric space is bounded. *Hint:* use the finite cover.

7. **Theorem.** Any compact subset of a metric space is closed and bounded. (Prove this.)

   *Hint:* put together several previous results.
Heine-Borel Theorem. The following are equivalent, for a set $S \subset \mathbb{R}^n$:

1. Every sequence in $S$ has a subsequence converging to a point of $S$.
2. $S$ is closed and bounded.
3. $S$ is compact: every open cover has a finite subcover.

1. Proof. We will prove that $(3) \implies (2) \implies (1) \implies (3)$. This will prove that all three criteria are equivalent, because ____________.

2. (3) $\implies$ (2). We will prove the contrapositive: if $S$ is not closed or not bounded, then there is some open cover that has no finite subcover. (Part 1) Suppose that $S$ is not closed. Then some convergent sequence of points from $S$ converges to a point $a$ that is not in $S$, because (a) ___________. Then $a$ is an accumulation point for $S$, because (b) ___________. Then the open cover $\{x : |x - a| > 1/n : n \in \mathbb{N}\}$ has no finite subcover, because (c) ___________. (Part 2) Suppose that $S$ is not bounded. Then the open cover $\{x : |x| > n : n \in \mathbb{N}\}$ has no finite subcover, because (d) ___________.

3. (2) $\implies$ (1). We will show that, if $S$ is closed and bounded, then every sequence in $S$ has a subsequence converging to a point of $S$. Take any sequence in $S \subset \mathbb{R}^n$. First, just look at the first of the $n$ components of each point. Since $S$ is bounded, the sequence of the first components is bounded, because (a) ___________. So for some subsequence, the first components converge, because (b) ___________. Similarly, for a further subsequence, the second components converge. Eventually, for some further subsequence, each of the components converge, because (c) ___________. The limit point is in $S$, because (d) ___________.

4. (1) $\implies$ (3). We will show that, if every sequence in $S$ has a subsequence converging to a point in $S$, then every open cover of $S$ has a finite subcover. First, we will show that every open cover has a countable subcover, and then we will show, using a proof by contradiction, that it must actually have a finite subcover. Given an open cover $\{G_\alpha\}$ of $S$, we will construct a countable subcover. Every point of $S$ lies in a ball of rational radius about a rational point, because (a) ___________. Each of these countably many balls is contained in some $G_\alpha$, because (b) ___________. So a countable open cover $\{V_i\}$ of $S$ is given by (c) ___________.

5. Now suppose that $\{V_i\}$ has no finite subcover. Choose $x_1 \in S \setminus V_1$. Choose $x_2 \in S \setminus (V_1 \cup V_2)$. Continue, choosing $x_n$ in $S \setminus \bigcup_{i=1}^n V_i$, which is always possible, because (a) ___________. For each value of $i$, there are only finitely many $x_n$, with $n < i$, contained in $V_i$, because (b) ___________. Because (c) ___________, the sequence $x_n$ has a subsequence converging to some $x \in S$, with $x \in V_i$ for some $i$. Thus infinitely many $x_n$ are contained in $V_i$, which is a contradiction because (d) ___________. Thus every open cover of $S$ has a finite subcover, as desired.
6. Prove that the union of two compact sets is compact, using:
   (a) Criterion (2)   (b) Criterion (1)   (c) Criterion (3)

7. Prove or give a counterexample: An arbitrary union of compact sets is compact.
1. \( S = [0, \infty) \) is a subset of the metric space \( \mathbb{R} \), but \( S \) itself is also a metric space, with the inherited distance metric from \( \mathbb{R} \). Which of the following are open sets in \( S \)?

(a) \((0, 1)\)  
(b) \([0, 1)\)  
(c) \((0, 1]\)  
(d) \([0, 1]\)

More on open sets. So far, we have used the “every point is contained in an open ball” characterization of an open set. Now we will have an alternative characterization.

2. **Theorem.** Let \( E \) be a metric space, and let \( S \) be a subset of \( E \), considered as a metric space itself. For a subset \( A \subset S \), the following are equivalent:

(1) \( A \) is open in \( S \).

(2) There exists a set \( \tilde{A} \) that is open in \( E \), such that \( A = \tilde{A} \cap S \)

Prove this. **Hint:** To prove that (1) \( \implies \) (2), show that \( \tilde{A} \cap S \subset A \) and that \( A \subset \tilde{A} \cap S \).

3. **Theorem.** Suppose \( K \subset S \subset E \). Then \( K \) is compact in \( S \) iff \( K \) is compact in \( E \).

**Proof.** \((\Leftarrow\Rightarrow)\) Suppose that \( K \) is compact in \( E \). We will show that \( K \) is also compact in \( S \), by showing that every open cover of \( K \) in \( S \) has a finite subcover. Take any open cover \( \{U_\alpha\} \) of \( K \) in \( S \). By the previous Theorem, for each \( \alpha \), \( U_\alpha = \tilde{U}_\alpha \) for a set \( \tilde{U}_\alpha \cap E \) that is open in \( E \). Then \( \{\tilde{U}_\alpha\} \) gives an open cover of \( S \) in \( E \). Since \( K \) is compact in \( E \), there exists a finite subcover of \( K \) in \( E \), so . . . (Finish the proof. Then prove the other direction.)

Connected sets. A metric space \( E \) is **connected** if the only subsets of \( E \) that are both open and closed are \( E \) and \( \emptyset \). A subset \( S \subset E \) is connected if it is connected when considered as a metric space.

4. Give an example of a subset that is connected, and a subset that is not connected, in:

(a) \( \mathbb{R} \)  
(b) \( \mathbb{R}^2 \)  
(c) \( \mathbb{R} \) with the discrete metric: \( d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases} \)

5. Metric spaces that are not connected.

(a) Use negation to write the definition: A metric space \( E \) is **not connected** if . . .

(b) Explain why this is equivalent to: A metric space \( E \) is **not connected** if there are disjoint, nonempty open sets \( A \) and \( B \) such that \( E = A \cup B \).

Totally disconnected. A set \( S \) is **totally disconnected** if it has at least two points, and if for all distinct points \( p_1, p_2 \in S \), the set \( S \) can be separated by disjoint open sets \( U_1 \) and \( U_2 \) into two pieces \( S \cap U_1 \) and \( S \cap U_2 \) containing \( p_1 \) and \( p_2 \), respectively. **Hint:** draw a picture

6. Show that the following sets are totally disconnected: \( \text{(a) } \mathbb{Z} \quad \text{(b) } \mathbb{Q} \)

Other bases. The **decimal expansion** of a number between 0 and 1 tells, in each decimal place, the number of \( 1/10^k \)'s, \( 1/10^2 \)'s, \( 1/10^3 \)'s, etc. needed to sum to the number, using digits between 0 and 9. The **binary expansion** and **ternary expansion** do the same, with the number of powers of \( 1/2 \) and \( 1/3 \), respectively, using digits \( \{0, 1\} \) and \( \{0, 1, 2\} \), respectively.

7. Write \( 3/8, 7/16 \) and \( 1/3 \) in binary. Write \( 5/9, 8/27 \) and \( 1/2 \) in ternary.

8. Guess: Is it possible for a subset of \( \mathbb{R} \) to be uncountable, closed, and totally disconnected?
1. Find the set of accumulation points in $\mathbb{R}^2$ for each of the following sets:
(a) $\{(p,q) : p,q \in \mathbb{Q}\}$
(b) $\{(m/n, 1/n) : m,n \in \mathbb{Z}, n \neq 0\}$

2. Give an example of a totally disconnected set $S \subset [0,1]$ whose closure is $[0,1]$.

The Cantor set. Start with the closed unit interval $[0,1]$. Remove the open middle third $(1/3, 2/3)$, leaving two closed intervals of length $1/3$. Remove the open middle third of each of these, leaving four closed intervals of length $1/9$. Continue. At the $n^{th}$ step, you have a set $S_n$ consisting of $2^n$ closed intervals each of length $1/3^n$. Let $C = \bigcap S_n$.

3. Draw $S_n$ for $n = 0, 1, 2, 3, 4$. $S_0$ should take up the entire width of your page.

4. Find the total length of $S_n$ as a function of $n$, and the total length of $C$.

5. Prove, in two sentences, that $C$ is compact.

6. (Problem from Rick Parris) Show that $1/4 \in C$. *Hint:* Use the self-similarity, or fractal structure, of the Cantor set, to show that $1/4$ is never removed.

7. Explain why the set $C$ is totally disconnected.

Dense and nowhere dense sets. Let $S$ be a subset of a metric space $E$. We say that $S$ is dense in $E$ if $\overline{S} = E$. Also see, and feel free to use, the equivalent definition of dense in Page P5 #4 part (2). We say that $S$ is nowhere dense in $E$ if, for every nonempty open set $U$ in $E$, $\overline{S} \cap U \neq U$.

8. Which of the following sets are dense in $\mathbb{R}$? Which are nowhere dense in $\mathbb{R}$?
(a) $[0,1]$ (b) $\mathbb{Q}$ (c) $\mathbb{Z}$ (d) the irrationals (e) $\mathbb{Q} \setminus (0,1)$
(f) rationals with a power of 2 in the denominator (g) the Cantor set

9. (Continuation) Are “dense” and “nowhere dense” opposites?

10. Do you think it is possible to define a continuous map $f : C \to [0,1]$? Do so if you can.
1. **Proposition.** Any closed subset of a compact set is compact.

*Proof.* Let $X$ be a compact subset of a metric space $E$, and let $S$ be a closed subset of $S$. We will show that $S$ is compact, by showing that (a) _______________. Let $\{G_\alpha\}$ be an open cover of $S$. We also know that $S^C$ is open, because (.) ______________. b. Then the union $\{G_\alpha\} \cup S^C$ give an open cover of (c) ______________. Since $X$ is compact... (d) (complete the proof).

2. **Proposition.** The Cantor set $C$ is uncountable.

*Proof.* Represent elements of $[0,1]$ as decimals in base 3, like $0.02102012\ldots$. At the first step in the construction of the Cantor set, we removed decimals with a 1 in the first decimal place, because (a) ______________. At the second step, we removed remaining decimals with a 1 in the second decimal place, and so on. Thus the Cantor set consists of all of the base 3 decimals consisting just of 0s and 2s, like $0.02202020\ldots$, because (b) ______________.

(c) Finish the proof by showing that the set of such decimals is uncountable.

3. Show that every point of the Cantor set is an accumulation point of the Cantor set. (Such sets are called perfect. Every nonempty perfect real set is uncountable.)

*Hint:* For any point $a \in C$, construct a sequence $a_n$ converging to $a$, with each $a_n \in S_n$.

4. Prove that the following are equivalent, for a subset $S$ of a metric space $E$:

(1) $S$ is dense in $E$, i.e. $\overline{S} = E$.

(2) Every ball about every point of $E$ contains a point of $S$.

5. Prove or give a counterexample: The union of two connected subsets of a metric space is connected.

6. **Proposition.** Let $\{S_i\}$ be a collection of connected subsets of a metric space $E$. Suppose that there exists $k$ such that, for all $i$, $S_i \cap S_k \neq \emptyset$. Then $\bigcup_i S_i$ is connected.

(a) Draw a picture for the case $E = \mathbb{R}^2$ to get intuition about why this statement is true.

(b) *Proof.* We will show that $\bigcup S_i$ is connected using a proof by contradiction. Suppose that $\bigcup S_i$ is not connected. Then $\bigcup S_i$ is the union of two disjoint open sets $A, B$. We will show that $A$ or $B$ must be empty. (Do this.)

*Hint:* write $S_k = (S_k \cap A) \cup (S_k \cap B)$ and use the fact that $S_k$ is connected.
Real Analysis

Continuity. “Nearby points are sent to nearby points.” There are three (!) equivalent definitions of what it means for a function to be continuous. We will explore each of them. Then we will prove their equivalence in Page 15 # 3.

Let \( E, E' \) be metric spaces with distance metrics \( d, d' \) respectively. Let \( f : E \to E' \) be a function, and let \( x_0 \in E \). We say that \( f \) is continuous at \( x_0 \) if:

1. for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that, for \( x \in E \), \( d(x, x_0) < \delta \implies d'(f(x), f(x_0)) < \epsilon \).
2. for every sequence \( \{x_n\} \) with \( \lim_{n \to \infty} x_n = x_0 \), we have \( \lim_{n \to \infty} f(x_n) = f(x_0) \).
3. for every open set \( U \subset E' \), \( f^{-1}(U) \) is open in \( E \).

We say that \( f \) is continuous if it is continuous everywhere in \( E \).

1. Let’s explore the epsilon-delta definition of continuity. For each function, compute \( f(x_0) \) and draw a sketch of \( f(x) \) on the interval \([x_0 - 1, x_0 + 1]\). In the next columns, write out the “allowable output range” \((f(x_0) - \epsilon, f(x_0) + \epsilon)\), and then write out the corresponding “permissible input range” \((x_0 - \delta, x_0 + \delta)\), if it exists, for each value of \( \epsilon \). Finally, decide if \( f \) is continuous at \( x_0 \).

<table>
<thead>
<tr>
<th>function and ( x_0 ) value</th>
<th>( f(x_0) )</th>
<th>sketch</th>
<th>( \epsilon = 1 )</th>
<th>( \epsilon = 0.1 )</th>
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<tr>
<td>( f_1(x) =</td>
<td>x</td>
<td>), ( x_0 = 0 )</td>
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<td>( f_2(x) = \begin{cases} \sin(1/x) &amp; x \neq 0 \ 0 &amp; x = 0 \end{cases} ), ( x_0 = 0 )</td>
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<td>( f_3(x) = \begin{cases} x &amp; x \leq 1 \ 2x - 0.99 &amp; x &gt; 1 \end{cases} ), ( x_0 = 1 )</td>
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<td>( f_4(x) = \begin{cases} x \cdot \sin(1/x) &amp; x \neq 0 \ 0 &amp; x = 0 \end{cases} ), ( x_0 = 0 )</td>
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<tr>
<td>( f_5(x) = \begin{cases} 1/x &amp; x \neq 0 \ 2 &amp; x = 0 \end{cases} ), ( x_0 = 0 )</td>
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<tr>
<td>( f_6(x) = \begin{cases} 1 &amp; x \in \mathbb{Q} \ 0 &amp; x \notin \mathbb{Q} \end{cases} ), ( x_0 = 0 )</td>
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2. When you couldn’t find a \( \delta \) for a particular \( \epsilon \), why not? When you could find a \( \delta \) for a particular \( \epsilon \), was \( \delta \) unique? If not, could you find a maximum value for \( \delta \)? A minimum?

3. Use the sequence definition to show that \( f_3(x) \) is not continuous.

Sets that are connected and not connected. A subset \( S \) of a metric space \( E \) is not connected if it can be separated by two disjoint open sets \( U_1 \) and \( U_2 \) into two nonempty pieces \( S \cap U_1 \) and \( S \cap U_2 \), such that \( S = (S \cap U_1) \cup (S \cap U_2) \). Otherwise, it is connected.

4. Prove, in two sentences, that any subset of \( \mathbb{R} \) that contains two distinct points \( a \) and \( b \), and does not contain all of the points between \( a \) and \( b \), is not connected.

5. Prove that an interval of real numbers is connected (perhaps by contradiction.)

April 2018 14 Diana Davis
Real Analysis

1. The open set definition of continuity uses inverse images. Let’s think about images. Prove or give a counterexample: If \( f : E \to E' \) is a continuous function, and \( \mathcal{U} \) is open in \( E \), then \( f(\mathcal{U}) \) is open in \( E' \).

2. Let \( g : E \to E' \) and \( f : E' \to E'' \) be continuous functions. Prove that their composition \( f \circ g \) is continuous, using each definition of continuity:
   (a) epsilon-delta definition
   (b) sequence definition
   (c) open set definition.
   (d) Which way did you most prefer? Which did you least prefer?

3. Theorem. The three definitions (1), (2), (3) of continuity are equivalent.
   (a) \( (1) \iff (2) \). Hint: Prove \( (1) \implies (2) \) directly, and \( (2) \implies (1) \) using the contrapositive.
   (b) \( (1) \implies (3) \): Let \( \mathcal{U} \) be an open set in \( E' \). We wish to show that \( f^{-1}(\mathcal{U}) \) is open, so we need to show that, for any \( p \in f^{-1}(\mathcal{U}) \), there is an open ball about \( p \) in the set (a). Let \( p \in f^{-1}(\mathcal{U}) \). Then \( f(p) \in \mathcal{U} \), so there exists \( \epsilon > 0 \) such that \( f(p) \) is contained in (b) in the set (c). Since we assume (1), we can choose \( \delta > 0 \) such that
   \[
   |x - p| < \delta \implies |f(x) - f(p)| < \epsilon, \text{ and thus } |x - p| \leq \delta/2 \implies |f(x) - f(p)| < \epsilon.
   \]
   Here we divided \( \delta \) by 2 so that (d).
   Now we have shown that \( B(p, \delta/2) \subseteq f^{-1}(B(f(p), \epsilon)) \subseteq f^{-1}(\mathcal{U}) \), so (e) (finish the proof).
   (3) \implies (1): \text{ Hint: Since the inverse image of the open ball about } f(p) \text{ of radius } \epsilon \text{ is open and contains } p, \text{ it contains some ball } B(p, \delta), \text{ so } |x - p| < \delta \implies |f(x) - f(p)| < \epsilon. \text{ Fill in the details.}

4. We (usually) do calculus in \( \mathbb{R}^n \). You may wonder, why not on some other metric space? Explain why the Intermediate Value Theorem does not hold for a function \( f : \mathbb{Q} \to \mathbb{R} \). Begin your answer by stating the IVT and drawing a picture to illustrate it.

Closed, complete and compact. We will explore relationships between these concepts.

5. Consider the set \( A = [0, 1] \cap \mathbb{Q} \) in the metric space \( E = \mathbb{Q} \).
   (a) Show that \( A \) is closed in \( E \).
   (b) Show that \( A \) is not complete.
   (c) Is \( A \) compact? Prove your answer correct (Hint: use a previous result).

6. Let \( A \) be a subset of a metric space \( E \). Show that if \( A \) is complete, then \( A \) is closed.

7. Let \( A \) be a subset of a complete metric space \( E \). Show that \( A \) is complete if and only if \( A \) is closed.
Real Analysis

Review for Midterm 1

1. Write the definition(s) for each term:
   (a) subset       (b) countable       (c) supremum       (d) minimum       (e) open
   (f) closed       (g) metric space     (h) bounded       (i) limit        (j) converge
   (k) subsequence  (l) lim inf         (m) monotone       (n) accumulation point
   (o) isolated point (p) boundary      (q) interior       (r) closure       (s) Cauchy
   (t) complete     (u) cover           (v) compact        (w) connected     (x) dense

2. For each characteristic, say whether a finite union and an arbitrary union of sets with that characteristic must still have it. If not, give a counterexample.

   Finite union?   Arbitrary union?

   (a) finite
   (b) countable
   (c) uncountable
   (d) open
   (e) closed
   (f) compact
   (g) bounded

3. Let $A$ be a subset of a metric space $E$.
   (a) Define the boundary of $A$.
   (b) Define what it means for $A$ to be open.
   (c) Prove that $A$ is open if and only if $A$ contains none of its boundary, i.e. $A \cap \partial A = \emptyset$.

4. For each of the following, say whether it is True or False. For those that are false, give a counterexample.
   (a) Every union of infinitely many countable sets is countable.
   (b) Every convergent sequence is Cauchy.
   (c) Every intersection of countably many open sets is open.
   (d) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.
   (e) Every compact metric space is complete.

5. For each term in problem 1, write down a result (Theorem, etc.) that uses it.

6. Do this: Circle the problems on this page, and note down any other problems, that you would like to discuss in class.
Real Analysis

1. Reflect on the course so far this semester. Write several sentences about:

(a) Things we are doing well. What should we continue to do?

(b) Things we can improve. What should we change, to make things better?

2. Show that the function $f(x) = 2x + 1$ is continuous at $x = 1/2$ by explicitly finding a $\delta$ for each of the following values of $\epsilon$: 
   
   (a) $\epsilon = 1$  
   (b) $\epsilon = 1/3$  
   (c) $\epsilon = \epsilon$

3. **Theorem.** The continuous image of a compact set is compact.

   *Proof.* Let $f : E \to E'$ be a continuous function, and let $E$ be compact. We will show that its image $f(E)$ is compact by showing that, given any open cover of $f(E)$, we can construct a finite subcover. Let $\{U_\alpha\}$ be an open cover of $f(E)$. Since each set $U$ in $\{U_\alpha\}$ is open in $E'$, each $f^{-1}(U)$ is open in $E$, because (a)_________ . Consider the set $S = \{f^{-1}(U) : U \in U_\alpha\}$. We claim that $S$ is an open cover of $E$. To see this, take any $p \in E$. Then $f(p) \in U$ for some $U \in U_\alpha$, so $p \in f^{-1}(U)$, so $S$ covers $E$. Since $E$ is compact, there is a finite subcover of $S$ covering $E$, i.e. $E \subset f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_k)$. Thus $f(E) \subset U_1 \cup \cdots \cup U_k$. So for any open cover $U_\alpha$ of $f(E)$, we have constructed a finite subcover (b)_________, so $f(E)$ is compact, as desired.

**Uniform continuity.** $f : E \to E'$ is uniformly continuous if, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $p, q \in E$,

$$d(p, q) < \delta \implies d'(f(p), f(q)) < \epsilon.$$

4. Let $E = (0, 1)$, and let $f_1(x) = 2x$ and $f_2(x) = 1/x$, functions from $E$ to $\mathbb{R}$.

   (a) Let $x_0 = 1/2$. Given any $\epsilon > 0$, find $\delta_1$ so that $|x - x_0| < \delta_1 \implies |f_1(x) - f_1(x_0)| < \epsilon$.

   (b) Let $x_0 = 1/2$. Given any $\epsilon > 0$, find $\delta_2$ so that $|x - x_0| < \delta_2 \implies |f_2(x) - f_2(x_0)| < \epsilon$.

   (c) Repeat part (b), for $x_0 = 1/10$.

   (d) Repeat part (c), for $x_0 = 1/10$.

   (e) Both functions are continuous on $(0, 1)$, but only one is uniformly continuous on $(0, 1)$. Explain geometrically what causes the difference. *Hint: draw a picture.*

5. **The Cantor function.** Define $f : [0, 1] \to [0, 1]$ as follows: at 0 it is 0, and at 1 it is 1. On the middle third of the interval, $f = 1/2$. On the middle thirds of the two remaining intervals, define $f$ to be $1/4$ and $3/4$. On the middle thirds of the remaining four intervals, define $f$ to be $1/8, 3/8, 5/8, 7/8$. Continue. $f$ extends to a continuous function on $[0, 1]$.

   (a) Make a large, accurate sketch of $f$. Such a function is called a devil’s staircase.

   (b) Define a function on the Cantor set as follows: For any $p \in [0, 1]$, express it in ternary, giving an infinite decimal of 0s and 2s. Divide it by 2, yielding 0s and 1s. Interpret this number in binary; this is $f(p)$. How is this function related to the Cantor function? Is it onto? Is it continuous?

6. Graph the following functions, for $n = 1, 2, 3, 4, 5$. $f_n(x) = \begin{cases} 
1 - nx & \text{if } 0 \leq x < 1/n \\
0 & \text{if } 1/n \leq x.
\end{cases}$

Is there a “limit function” as $n \to \infty$? If so, graph that, too.
Real Analysis

Bounded functions. $f : E \to \mathbb{R}$ is bounded if there exists $M > 0$ such that $|f(p)| < M$ for all $p \in E$. In general, $f : E \to E'$ is bounded if there exists $M > 0$ and $p' \in E'$ such that $f(E) \subset B_M(p')$.

1. What is $p'$ for the first definition, for a function $f : E \to \mathbb{R}$?

2. Corollary to Page 17 # 2. If $f : E \to E'$ is continuous, and $E$ is compact, then $f$ is bounded. (Prove this.) Hint: A compact set is closed and bounded in any metric space (see Page P4 # 5-6). The converse is false in general, though true in $\mathbb{R}^n$ as proved in the Heine-Borel Theorem.

3. (Continuation) Show that it is necessary for $E$ to be compact, by giving an example of an unbounded continuous function on a non-compact metric space.

Sequences of functions. Let $f_n : E \to E'$ and let $p \in E$. The sequence $f_1, f_2, \ldots$ converges at $p$ if $f_1(p), f_2(p), \ldots$ converges as a sequence of points in $E'$. The sequence $f_1, f_2, \ldots$ converges (on $E$) if it converges at every point in $E$. If $f_1, f_2, \ldots$ converges, we define the limit function to be $f(p) = \lim_{n \to \infty} f_n(p)$ for each $p \in E$.

4. Say whether the following sequences converge, and if so, to what limit function:
   (a) $g_1(x) = 1, g_2(x) = 1 + x, g_3(x) = 1 + x + \frac{x^2}{2}, \ldots, g_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n-1}}{(n-1)!}$
   (b) $h_1(x) = x, h_2(x) = x^2, \ldots, h_n(x) = x^n$, on $[0, 1]$. This is on the cover of both textbooks.

5. Explain why uniformly continuous $\implies$ continuous. Explain why continuous $\not\implies$ uniformly continuous by providing a counterexample.

6. In fact, continuous $\implies$ uniformly continuous in the special case when $E$ is compact. Theorem. Let $f : E \to E'$ be a continuous function. If $E$ is compact, then $f$ is uniformly continuous.

Proof. Given any $\epsilon > 0$, we will construct a $\delta$ such that $d(p,q) < \delta \implies (a)$. Given $\epsilon > 0$, we know that for each $x_0 \in E$, there is a $\delta_{x_0} > 0$ such that $d(x,x_0) < \delta_{x_0} \implies d'(f(x),f(x_0)) < \epsilon/2$, because (c). Consider the open ball $U_{x_0} = \{x : d(x,x_0) < \delta_{x_0}/2\}$. The collection $\{U_x\}$ of all such open balls covers $E$, because (d). Since $E$ is compact, it has a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$. Let $\delta = \min\{\delta_{x_i}/2 : i = 1, \ldots, n\}$. We will show that this is the $\delta$ with the desired property.

Suppose that $d(x,x_0) < \delta$. Since $x_0 \in E$, $x_0 \in U_{x_j}$ for some $j$ in $\{1, \ldots, n\}$, so $d(x_0, x_j) < \delta_{x_j}/2$. Since $d(x,x_0) < \delta \leq \delta_{x_j}/2$, we have $d(x,x_j) < \delta_{x_j}$, by (d). Therefore, $d'(f(x_0), f(x_j)) < \epsilon/2$ and $d'(f(x), f(x_j)) < \epsilon/2$, so $d'(f(x), f(x_0)) \leq d'(f(x_0), f(x_j)) + d'(f(x), f(x_j)) < \epsilon/2 + \epsilon/2 = \epsilon$, as desired.
1. **Corollary** to Page 17 # 2. A continuous real-valued function on a nonempty compact metric space attains a maximum and minimum. (This is essential for calculus!)

Proof. Let \( f : E \to \mathbf{R} \) be a continuous function on a nonempty compact metric space \( E \). Then \( f(E) \) is closed and bounded because (a) __________ and nonempty because (b) __________. A nonempty bounded set has an infimum \( a \) and a supremum \( b \), by Page __ # __. Furthermore, \( a \) and \( b \) are accumulation points of \( f(E) \), so there are sequences in \( f(E) \) converging to \( a \) and \( b \). Since \( f(E) \) is closed, the limits of these sequences are in \( f(E) \), because (d) __________. Thus \( a, b \in f(E) \), so \( f(E) \) attains a maximum and minimum.

2. Give a counterexample to the following statement: If \( f : \mathbf{R} \to \mathbf{R} \) is continuous and \( S \) is connected, then \( f^{-1}(S) \) is connected.

3. Give an example of a continuous, bounded function on \((0, 1)\) that is not uniformly continuous. **Hint**: One option is a “sawtooth” where the teeth get narrower and steeper.

4. For each of the following functions (also used in the next problems) from \([0, 1]\) to \([0, 1]\), give the limit function as \( n \to \infty \), and say whether it is continuous. **Hint**: draw a picture

(a) \( f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \leq x < 1/n \\ 0 & \text{if } 1/n \leq x \end{cases} \)

(b) \( g_n(x) = x/n \)

(c) \( h_n(x) = x^n \) (sequence of functions on the cover of both textbooks)

It is a bit disturbing that, in some cases, a limit of continuous functions is not continuous. The problem is that although each \( f_n \to f \) at each point, for some functions, some points take longer to converge than others. We can express this difference precisely:

**Pointwise convergence.** A sequence of functions \( f_n : E \to E' \) converges pointwise to \( f \) on \( E \) if, for every \( x \in E \), and given any \( \epsilon > 0 \), there exists an \( N \) such that

\[ n > N \implies d'(f_n(x), f(x)) < \epsilon. \]

Notice that in this definition, \( N \) depends on both \( x \) and \( \epsilon \).

5. For the given \( x \) and \( \epsilon \), find \( N \) such that \( n > N \implies |g_n(x) - f(x)| < \epsilon. \)

(a) \( x = 1/2, \epsilon = 0.01 \)    (b) \( x = 0.9, \epsilon = 0.01 \)

6. Repeat problem 5 for \( h_n(x) \).

So far, we have mostly considered metric spaces consisting of points, such as \( \mathbf{R}, \mathbf{R}^2 \) and the discrete metric space. Now let’s consider a metric space consisting of functions.

**Spaces of functions.** Let \( \mathcal{C}(E, E') \) be the set of all continuous functions from the metric space \( E \) to the metric space \( E' \). Here \( \mathcal{C} \) stands for continuous. In the special case where \( E' = \mathbf{R} \), we call the set \( \mathcal{C}(E) \). It is a metric space (Page P7 # 2), under the distance metric

\[ D(f, g) = \max\{d'(f(p), g(p)) : p \in E\}, \]

where as usual \( d' \) is the distance metric in \( E' \). Notice that for this metric to be well defined, the maximum must be finite.

7. For each of the following, \( f, g : \mathbf{R} \to \mathbf{R} \), find \( D(f, g) \). **Hint**: draw a picture

(a) \( f(x) = \sin(x), g(x) = 0 \)    (b) \( f(x) = x + \sin(x), g(x) = x \)
Real Analysis

Warm-up problems

1. Sometimes, when we wish to prove an “if and only if” statement, we can complete both the “⇒” and “⇐” directions of the proof at the same time. The idea is that we could prove each direction separately, but the reasoning would be the same in both directions, because all of the steps are actually equivalences, not implications. Use this proof strategy to prove the following statement: A set $S$ is closed if and only if, for every point $p$ in $S^C$, there is a ball about $p$ completely contained in $S^C$.

2. (Inspired by Midterm 1) Prove that the set of subsets of an infinite set is infinite.

Some things to prove

3. Proposition. $C(E, E')$ with the metric $D$ is a metric space.

Prove this, by proving each of the four requirements for a metric space.

4. Theorem. Let $f : E \to E'$ be continuous. If $E$ is connected, then $f(E)$ is connected. (Prove this.) Hint: perhaps by contradiction.

Intermediate Value Theorem in R. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then $f$ attains all values between $f(a)$ and $f(b)$.

Intermediate Value Theorem in general: Let $f : E \to \mathbb{R}$ be continuous, let $E$ be connected, and let $p_1, p_2 \in E$. Then $f$ attains all values between $f(p_1)$ and $f(p_2)$.


6. Consider the statement: If $f_n \to f$ uniformly, then $f_n^2 \to f^2$ uniformly.

(a) Give a counterexample to the statement.

(b) Add a simple hypothesis, and prove the revised statement.
1. Prove that your height in inches once equaled your weight in pounds.

Uniform convergence. A sequence of functions \( f_n : E \to E' \) converges uniformly to \( f \) on \( E \) if, given any \( \epsilon > 0 \), there exists an \( N \) such that

\[
 n > N \implies d'(f_n(x), f(x)) < \epsilon
\]

for all \( x \in E \).

2. Explain the difference between the definition of pointwise convergence and the definition of uniform convergence. In Page 19 #4, which functions converge uniformly?

We saw in Page 19 #4 that some limits of continuous functions are not continuous. For the limit to be continuous, uniform convergence is exactly what we need.

3. Theorem. A uniform limit of continuous functions is continuous.

Proof. We will show that, for a uniformly convergent sequence of functions \( f_1, f_2, \ldots \), with \( f_n : E \to E' \) for each \( n \), \( \lim_{n \to \infty} f_n(x) = f(x) \) is continuous. More precisely, we will show that, given any \( \epsilon > 0 \), we can find \( \delta \) such that (a) \( \delta \).

The idea is that because the sequence of functions converges uniformly, we can handle all points \( x \) near any particular point \( p \) by looking at one \( f_n \) (specifically \( f_{N+1} \)) that is uniformly near the limit function \( f \).

Given any \( \epsilon > 0 \), choose \( N \) such that for all \( x \in E \),

\[
 n > N \implies d(f_n(x), f(x)) < \epsilon/3 \quad (1)
\]

We can find such an \( N \) because (b) \( \delta \).

Since \( f_{N+1} \) is continuous, given any \( p \in E \), we can choose \( \delta > 0 \) such that

\[
 d(x, p) < \delta \implies d'(f_{N+1}(x) - f_{N+1}(p)) < \epsilon/3. \quad (2)
\]

We know that \( f_{N+1} \) is continuous because (c) \( \delta \).

We can find such a \( \delta \) because (d) \( \delta \).

So if \( d(x, p) < \delta \), we have

\[
 d'(f(x), f(p)) \leq d'(f(x), f_{N+1}(x)) + d'(f_{N+1}(x), f_{N+1}(p)) + d'(f_{N+1}(p), f(p)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,
\]

as desired. Here the inequality in the first line is true by (e) \( \delta \).

The first term on the right hand side is less than \( \epsilon/3 \) by equation (f) \( \delta \), the second term on the right hand side is less than \( \epsilon/3 \) by equation (g) \( \delta \), and the third term on the right hand side is less than \( \epsilon/3 \) by equation (h) \( \delta \).

4. Consider continuous functions \( f_n : [0, 1] \to \mathbb{R} \) such that, for each \( x \in [0, 1] \),

\[
 0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots,
\]

converging pointwise to \( f(x) \). Must \( f \) be continuous? Prove it or give a counterexample.

5. Consider a differentiable function \( f : \mathbb{R} \to \mathbb{R} \), as in single-variable calculus. Write the equation of the tangent line to the graph of \( y = f(x) \) at the point \( (x_0, f(x_0)) \).
Real Analysis

For a sequence of points, we wanted a way to discuss convergence without knowing what the limit is. This notion is “Cauchy” — there exists $N$ such that $n, m > N \implies d(a_n, a_m) < \epsilon$. We would like to have an analogous way to talk about a sequence of functions converging, without knowing what the limit is. This notion is uniformly Cauchy, combining the notion of the sequence of functions converging uniformly, with the notion of the function values being a Cauchy sequence at each point.

1. **Theorem.** A sequence of functions mapping to a complete metric space is uniformly convergent if and only if the sequence of functions is uniformly Cauchy.

   **Proof.** We will prove that, for a sequence of functions $f_n : E \to E'$, if $E'$ is complete, then $f_n$ is uniformly convergent if and only if, for all $\epsilon > 0$, there exists $N$ such that

   $n, m > N \implies d'(f_n(p), f_m(p)) < \epsilon$ for all $p \in E$, or in other words, the sequence of $f_n$ is uniformly Cauchy.

   **Proof.** ($\Rightarrow$) We will assume that $f_n \to f$ uniformly, and show that $f_n$ is uniformly Cauchy. Suppose that $f_n$ converges uniformly to $f$. Then, given $\epsilon > 0$, there exists $N > 0$ such that $d'(f(x), f_n(x)) < \epsilon/2$ for all $n > N$ and all $x \in E$, because (a) \[\] So if $n, m > N$, we have

   \[ d'(f_n(x), f_m(x)) \leq d'(f_n(x), f(x)) + d'(f(x), f_m(x)) < \epsilon/2 + \epsilon/2 = \epsilon, \]

   as desired. Here the inequality in the first line is true by (b) \[\] Each of the terms on the right-hand side of the first line are less than $\epsilon/2$ because (c) \[\]

   ($\Leftarrow$) We will assume that $f_n$ is uniformly Cauchy, and show that $f_n \to f$ uniformly.

   Suppose that the sequence $f_n$ is uniformly Cauchy. Then for all $x \in E$, and for every $n$, \(\{f_n(x)\}\) is a Cauchy sequence in $E'$, because (d) \[\] Since $E'$ is complete, \(\{f_n(x)\}\) converges to a point in $E'$ because (e) \[\], and we will call this point $f(x)$. This shows that $f_n \to f$ pointwise. Now we need to show that $f_n \to f$ uniformly.

   Given any $\epsilon > 0$, choose $N$ such that

   \[ n, m > N \implies d'(f_n(x), f_m(x)) < \epsilon/2 \text{ for all } x \in E. \quad (1) \]

   We can find such an $N$ because (f) \[\] We want to show that, for any $n > N$ and any $x \in E$, $d'(f_n(x), f(x)) < \epsilon$.

   Fix a particular such choice of $n$ and $x$. Now look at the ball $B_{\epsilon/2}(f_n(x))$. The sequence $f_1(x), f_2(x), \ldots$ eventually enters this ball and stays within it, by (g) \[\]. So $f(x) \in B_{\epsilon/2}(f_n(x))$, because (h) \[\], Here we use the closure of the ball because (i) \[\]. Thus,

   \[ d'(f_n(x), f(x)) \leq \epsilon/2 < \epsilon, \]

   as desired. Here the first inequality is true because (j) \[\].
2. In single-variable calculus, to find $f'(x)$ at some point $x_0$, you took $h$ to be a small number (positive or negative) and took the limit:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

(a) Explain, using a picture, why this limit gives the instantaneous slope of $f$ at $x_0$.

(b) Compute the above limit (meaning: write it out, expand, do arithmetic, take the limit of the expression as $h \to 0$) for $f(x) = x^2$ and $x_0 = 1$. Is the answer what you expected?

3. In each of the following, $S \subset \mathbb{R}$ is a compact set, and $\mathcal{G}$ is an open cover of $S$. For each one, explain why $\mathcal{G}$ is an open cover of $S$, and give a finite subcover of $\mathcal{G}$ that covers $S$.

(a) $S = \{\pi\}, \mathcal{G} = \{(1/(n+1), n) : n \in \mathbb{N}\}$

(b) $S = [-3, 11], \mathcal{G} = \{(n, n + 2) : n \in \mathbb{Z}\}$

(c) $S = [0, 1], \mathcal{G} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$

4. Recall the $\epsilon - \delta$ definition of continuity (Page 14 #1). For each of the following statements, identify the logical error, and explain how the $\epsilon - \delta$ definition of continuity is violated. For each one, give an example that illustrates the error, and draw a picture.

(a) Let $f$ be a function, and let $x_0 = 2$. For $\epsilon > 0.2$, there always exists a $\delta$ such that

$$x \in (2 - \delta, 2 + \delta) \implies f(x) \in (f(2) - \epsilon, f(2) + \epsilon).$$

Therefore, $f$ is continuous at $x_0 = 2$.

(b) Let $g$ be a function, and let $x_0 = -3$. There exists an $\epsilon > 0$ such that, for every $\delta > 0$, we have

$$x \in (-3 - \delta, -3 + \delta) \implies g(x) \in (g(-3) - \epsilon, g(-3) + \epsilon).$$

Therefore, $g$ is continuous at $x_0 = -3$.

(c) Let $h$ be a function, and let $x_0 = -2$. For every $\delta > 0$, we can find an $\epsilon > 0$ such that

$$x \in (-2 - \delta, -2 + \delta) \implies h(x) \in (h(-2) - \epsilon, h(-2) + \epsilon).$$

Therefore, $h$ is continuous at $x_0 = -2$. 
Real Analysis

We have a metric space $C(E, E')$ (Page 19), where the “points” are actually functions.

1. **Proposition.** For a sequence $f_n \in C(E, E')$, $f_n \to f$ in $C(E, E')$ if and only if $f_n \to f$ uniformly on $E$.

   **Proof.** We will show that this is true by showing that the two statements are connected by a chain of equivalent statements (recall Page P7 # 1).

   $f_n \to f$ in $C(E, E') \iff$ for all $\epsilon > 0$, exists $N > 0$ such that $n > N \implies D(f_n, f) < \epsilon$ (1)
   $\iff \max\{d'(f_n(x), f(x)) : p \in E\} < \epsilon$ (2)
   $\iff d'(f_n(x), f(x)) < \epsilon$ for all $x \in E$ (3)
   $\iff f_n$ converges uniformly to $f$ on $E$. (4)

   Justify each of the four equivalences.

2. (Continuation) We have shown that the function $f_n(x) = x/n$, with $f : [0, 1] \to [0, 1]$, converges uniformly to $f(x) = 0$.

   (a) Use the previous Proposition to show that $f_n \to f$ in $C([0, 1], [0, 1])$.

   (b) Explain why it makes sense, geometrically and using the definition of convergence in $C$.

3. (Continuation) We have shown that the function $f_n(x) = x^n$, with $f : [0, 1] \to [0, 1]$, converges pointwise but not uniformly, to $f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 0. \end{cases}$.

   (a) Use the previous Proposition to show that $f$ does not converge to $f$ in $C([0, 1], [0, 1])$.

   (b) Explain why the convergence breaks down for this function.

**Differentiability.** In our continuing quest to put all of calculus on a rigorous mathematical basis, we will now study differentiability. Let $U \subset \mathbb{R}$ be an open set, let $f : U \to \mathbb{R}$, and let $x_0 \in U$. We say that $f$ is differentiable at $x_0$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

We denote this limit by $f'(x_0)$ and call it the derivative of $f$ at $x_0$.

4. Answer these questions about the definition of differentiability.

   (a) Why is the above definition equivalent to the “$x + h$” definition in Page 21 # 2 (when $U = \mathbb{R}$)? Use words, algebra and a picture to justify your answer.

   (b) Why must the domain and range of $\mathbb{R}$ be real numbers?

   (c) Why must $x_0$ be an element of an open set (or, why must the domain $U$ be open)?

5. A real-valued sequence $\{x_n\}$ can be thought of as a function $f : \mathbb{N} \to \mathbb{R}$. Prove or give a counterexample: Every sequence of real numbers is a continuous function.
Real Analysis

Warm-up problems

1. Show that the set of finite subsets of \( \mathbb{N} \) is countable.

2. Consider the “ruler” function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \).

Here we assume as usual that \( \frac{p}{q} \) is in lowest terms, \( p \in \mathbb{Z} \), \( q \in \mathbb{N} \). Sketch the function on \([0,1]\), and then explain why it is called the ruler function.

Some things to prove

3. Let \( f_n(x) = x/n \). Prove that \( f_n(x) \to f(x) = 0 \) uniformly on \([0,1]\).

Does \( f_n(x) \to 0 \) uniformly on \( \mathbb{R} \)?

**Theorem (Differentiation rules).** If \( f,g : \mathbb{R} \to \mathbb{R} \) are differentiable at \( x_0 \in \mathbb{R} \), then \( kf, f+g, \) and \( fg \) are also differentiable at \( x_0 \), as is \( f/g \) as long as \( g(x_0) \neq 0 \) and \( k \in \mathbb{R} \).

Furthermore,

\[
(kf)'(x_0) = k \cdot f'(x_0), \\
(f + g)'(x_0) = f'(x_0) + g'(x_0), \\
(fg)'(x_0) = f(x_0) \cdot g'(x_0) + g(x_0) \cdot f'(x_0), \text{ and} \\
(f/g)'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.
\]

We will prove the Product Rule, and omit proofs of the other three rules.

4. Prove the Product Rule: Show that the limit below is equal to the result given above.

**Hint:** add and subtract \( f(x) \cdot g(x_0) \) in the numerator, factor, and take the limit.

\[
(fg)'(x_0) = \lim_{x \to x_0} \frac{f(x) \cdot g(x) - f(x_0) \cdot g(x_0)}{x - x_0}
\]

**Note:** \TeX{} is a typesetting language. \LaTeX{} is a typesetting language, with labels. The idea is that you label things with a name, and then reference them by name. So for instance, if I had a Theorem 3 and referenced it by typing \texttt{Theorem 3}, then if I added a theorem above it and it became Theorem 4, I’d have to manually change 3 to 4. If I label it and refer to it by label, \LaTeX{} renumbers everything automatically. In writing up your proof for this problem, I expect that you will have a series of equalities, and you will want to refer to each line. To do this, you label the equation with a label e.g. \texttt{\label{firstline}} and then refer to it later with e.g. \texttt{\ref{firstline}} or \texttt{\eqref{firstline}}. Please use these in your writeup.

5. Prove that the ruler function is continuous at every irrational.

6. **Theorem.** If \( E \) is compact and \( E' \) is complete, then the metric space \( C(E,E') \) with metric \( D(f,g) = \max\{d'(f(x),g(x)) : x \in E\} \) is a complete metric space.

**Hint:** Combine several previous results.
1. **Theorem (Chain Rule).** Let $\mathcal{U}, \mathcal{V} \subset \mathbb{R}$, let $f : \mathcal{U} \to \mathcal{V}$, and let $g : \mathcal{V} \to \mathbb{R}$. Let $x_0 \in \mathcal{U}$, and suppose that $f'(x_0)$ and $g'(f(x_0))$ exist. Then $(g \circ f)'(x_0)$ exists, and

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

**Proof.** Since $f'(x_0)$ exists, by definition we know that $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. This limit exists if and only if, for all sequences $x_n \to x_0$, $\lim_{x \to x_0} \frac{f(x_n) - f(x_0)}{x_n - x_0}$ exists.

Since $f$ is differentiable, if we have a sequence $x_n \in \mathcal{U}$ with $x_n \to x_0$ and $x_i \neq x_0$ for all $i$, we have $f(x_1), f(x_2), \ldots \to f(x_0)$, because (a) \[ \lim_{x \to x_0} \frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0) \] and we also have

$$g'(f(x_0)) = \lim_{v \to f(x_0)} \frac{g(v) - g(f(x_0))}{v - f(x_0)}$$

because (b) \[ g'(f(x_0)) = \lim_{v \to f(x_0)} \frac{g(v) - g(f(x_0))}{v - f(x_0)} \]

Similarly, since $g$ is differentiable, if we have a sequence $v_n \in \mathcal{V}$ with $v_n \to f(x_0)$ and $v_i \neq f(x_0)$ for all $i$, we have $g(v_1), g(v_2), \ldots \to g(f(x_0))$, because (c) \[ \lim_{x \to x_0} \frac{g(v_n) - g(f(x_0))}{v_n - f(x_0)} = g'(f(x_0)), \] and we also have

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Thus, if we have $f(x_1), f(x_2), \ldots \in \mathcal{V}$, with $f(x_n) \to f(x_0)$, and $f(x_i) \neq f(x_0)$ for all $i$, then

$$\lim_{x \to x_0} \frac{g(f(x_n)) - g(f(x_0))}{f(x_n) - f(x_0)} = g'(f(x_0)).$$

In order to show that $g \circ f$ is differentiable at $x_0$ with derivative $g'(f(x_0)) \cdot f'(x_0)$, we need to show that if $x_n \to x_0$ in $\mathcal{U}$ with $x_n \neq x_0$, then \[ g(f(x_n)) - g(f(x_0)) \to g'(f(x_0)) \cdot f'(x_0). \]

It would be nice to obtain this by simply taking the product of the equations (1) and (2). However, we need $f(x_n) \neq f(x_0)$, because (d) \[ f(x_n) \neq f(x_0), \] \[ f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \]

**Finite case.** Suppose that there are only finitely many $n$ for which $f(x_n) = f(x_0)$. Then we can start the limit after the last such $n$, and say

$$\lim_{x \to x_0} \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} = \lim_{x \to x_0} \frac{f(x_n) - f(x_0)}{x_n - x_0} \cdot \frac{g(f(x_n)) - g(f(x_0))}{f(x_n) - f(x_0)} = g'(f(x_0)) \cdot f'(x_0),$$

as desired.

**Infinite case.** Suppose that there are infinitely many $n$ for which $f(x_n) = f(x_0)$. Then these $x_n$s form an infinite sequence $x_{n_k}$, for which $f(x_{n_k}) = f(x_0)$ for all $k$. Then \[ f(x_{n_k}) - f(x_0) = 0, \] so $f'(x_0) = 0$, because (e) \[ f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \]

Thus, \[ g'(f(x_0)) \cdot f'(x_0) = 0, \] so we need to prove that \[ \lim_{x \to x_0} \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} \to 0 \] in order to finish the result, because (f) \[ \lim_{x \to x_0} \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} \to 0. \] To do this, break the sequence $\{x_n\}$ into two subsequences, $\{x_{n_k}\}$, and the terms not in $\{x_{n_k}\}$. For $\{x_{n_k}\}$, the above quotient is zero, as shown. For the complementary subsequence, we use the proof as in the finite case to show that the limit is\[ g'(f(x_0)) \cdot f'(x_0), \] which is 0. This finishes the proof, because (g)\[ g'(f(x_0)) \cdot f'(x_0) = 0. \]
2. Prove that \( f(x) = x^2 \) is continuous on \( \mathbb{R} \), by finding a \( \delta \) for any given \( \epsilon \) and \( x \).

3. In this problem, we will justify the statement: \( f \) is differentiable at \( x_0 \) if and only if it is well approximated by its tangent line there.

(a) Write out the \( \epsilon - \delta \) version of the statement

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).
\]

There should be a hypothesis involving \( \epsilon, \delta, x \) and \( x_0 \), and a conclusion that is an inequality with a difference on one side and an \( \epsilon \) on the other.

(b) Rearrange your inequality so that it becomes the following:

\[
\left| f(x) - \left( f(x_0) + f'(x_0)(x - x_0) \right) \right| < \epsilon |x - x_0|
\]

(c) Explain how to interpret the above as saying that \( f(x) \) is well approximated by its tangent line at \( x_0 \), when \( x \) is close to \( x_0 \).

4. Proposition. For a sequence \( f_n \in C(E, E') \), \( f_n \) is Cauchy if and only if \( f_n \) is uniformly Cauchy.

Proof. Again, we will connect the two statements by a chain of equivalences:

\[
\{f_n\} \text{ is a Cauchy sequence} \\
\Leftrightarrow \text{for all } \epsilon > 0, \text{ exists } N \text{ such that } m, n > N \implies D(f_n, f_m) < \epsilon \quad (1) \\
\Leftrightarrow \text{for all } \epsilon > 0, \text{ exists } N \text{ such that } m, n > N \implies d'(f_n(x), f_m(x)) < \epsilon \text{ for all } x \in E \quad (2) \\
\Leftrightarrow \{f_n\} \text{ is uniformly Cauchy.} \quad (3)
\]

Justify each of the three equivalences.

5. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be differentiable at \( x_0 \). Justify each of the following (include a picture):

(a) If \( x_0 \) is a maximum for \( f \), then \( f(x) - f(x_0) \leq 0 \) for all \( x \) near \( x_0 \).

(b) If \( x_0 \) is a minimum for \( f \), then \( f(x) - f(x_0) \geq 0 \) for all \( x \) near \( x_0 \).

6. Consider the set of subsets of the natural numbers \( \mathbb{N} \). Show that this set is uncountable, perhaps by giving an explicit one-to-one correspondence with \( (0, 1] \). \( \text{Hint: binary} \)
1. **Proposition (finding critical points).** If a function is differentiable at an interior minimum or maximum point, then its derivative is 0 there.

This fact is the basis for finding maxima and minima in calculus: set the derivative equal to 0, and solve for $x$.

**Proof.** We will show that, for an open set $U \subset \mathbb{R}$ and a function $f : U \to \mathbb{R}$, if $f$ attains a maximum or minimum at $x_0 \in U$, and if $f$ is differentiable at $x_0$, then $f'(x_0) = 0$. We will prove the contrapositive, supposing that $f'(x_0) \neq 0$ and then showing that $x_0$ cannot be a maximum, nor a minimum.

Suppose that $|f'(x_0)| > 0$. We will use the number $|f'(x_0)|/2$ as our “$\varepsilon$” and apply the definition of differentiability: Because (a) $|f'(x_0)|/2$ is positive, there exists $\delta > 0$ such that, for $x \neq x_0$ and $|x - x_0| < \delta$, we have

$$\frac{|f'(x_0)|}{2} < \frac{f(x) - f(x_0)}{x - x_0} < f'(x_0) + \frac{|f'(x_0)|}{2}$$

Justify the (b) first, (c) second, and (d) third lines above. Now we have two cases:

**Positive case.** Suppose that $f'(x_0) > 0$. Then since $|f'(x_0)| = f'(x_0)$, equation (e) reduces to

$$\frac{1}{2} f'(x_0) < \frac{f(x) - f(x_0)}{x - x_0} < \frac{3}{2} f'(x_0),$$

so $\frac{f(x) - f(x_0)}{x - x_0}$ is always positive, because (f).

**Negative case.** Suppose that $f'(x_0) < 0$. Then since $|f'(x_0)| = -f'(x_0)$, equation (g) reduces to

$$\frac{3}{2} f'(x_0) < \frac{f(x) - f(x_0)}{x - x_0} < \frac{1}{2} f'(x_0),$$

so $\frac{f(x) - f(x_0)}{x - x_0}$ is always negative, because (h).

(i) Complete the proof.
2. Rolle’s Theorem. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$, and that $f(a) = f(b)$. Then for some $c \in (a, b)$, $f'(c) = 0$.

Rolle’s Theorem is the basis for the Mean Value Theorem, which is very important in calculus.

Proof. $f$ attains a maximum and a minimum on $[a, b]$, because (a) $f$ is continuous on $[a, b]$.

If the maximum occurs at some point $p$ on the interior, then by (b) $f'(p) = 0$, so let $c = p$ and we are done. The same argument holds for the minimum. If neither the maximum nor the minimum occurs on the interior, then they both occur at the endpoints, so (c) (finish the proof).

3. Proposition. If $f$ is differentiable at $x_0$, then $f$ is continuous at $x_0$.

Proof. We will show that, if $U$ is an open subset of $\mathbb{R}$, and $f : U \to \mathbb{R}$ is differentiable at $x_0 \in U$, then $f$ is continuous at $x_0$.

Since $f$ is differentiable at $x_0$, we know from (a) Page ___ # ___ that for any $\varepsilon > 0$, we can choose $\delta > 0$ so that

$$|x - x_0| < \delta \implies |f(x) - f(x_0) - f'(x_0)(x - x_0)| < \varepsilon|x - x_0|.$$  

Thus, $|x - x_0| < \delta$ implies

$$|f(x) - f(x_0)| \leq |f(x) - f(x_0) - f'(x_0)(x - x_0)| + |f'(x_0)(x - x_0)| \leq (\varepsilon + |f'(x_0)|) \cdot |x - x_0|.  \quad (1)$$

(2)

Justify the inequalities in the (b) first and (c) second lines. Now choose $\delta = \min \left\{ \delta, \frac{\varepsilon}{\varepsilon + |f'(x_0)|} \right\}$. (d) Use the above to show that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon,$$

as desired.

(e) Explain why we used the symbols $\delta$ and $\varepsilon$ at the beginning instead of $\delta$ and $\varepsilon$.

4. (Continuation) State the converse of the previous Proposition. Then either prove it or give a counterexample.

5. Is the set $\{(-1)^n/n : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ compact? Prove your answer correct, using the open cover definition of compactness.

6. Consider again the ruler function $f : \mathbb{R} \to \mathbb{R}$ defined in Page P8 # 2. At which points is $f$ differentiable?
1. The hypothesis for the preceding Rolle’s Theorem, and for the upcoming Mean Value Theorem, is that \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\).

(a) Why did we need \( f \) to be continuous on \([a, b]\) instead of just \((a, b)\)?

(b) Why don’t we ask for \( f \) to be differentiable on \([a, b]\) instead of just on \((a, b)\)?

2. The picture on the right is meant to illustrate the Mean Value Theorem. On the same axes, sketch the function \( g(x) = f(x) - x \). You will need to make reasonable assumptions about the axis scaling.

3. **Mean Value Theorem.** Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then for some \( c \in (a, b) \),

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

The MVT is essentially the same as Rolle’s Theorem, just “tilted,” or perhaps we could call it “vertically sheared.”

**Proof.** By horizontal scaling and translation, we may assume that \([a, b] = [0, 1]\), because (a). If \( f(0) = f(1) \), then we are done, because (b). If not, then by vertical scaling and translation, we may assume that \( f(0) = 0 \) and \( f(1) = 1 \), because (c). Let \( g(x) = f(x) - x \). (d) finish the proof

4. Check the Mean Value Theorem for the function \( f(x) = x^3 \) on \([0, 1]\). (This means: determine \( a, b, f(a) \), and \( f(b) \) in this case, and find the \( c \) that satisfies the equation above.)

The following result is equivalent to the Mean Value Theorem (MVT). In words, it says that “how a function varies on an interval depends on the length of the interval, multiplied by some bound on the derivative in that interval.”

5. **Corollary 1 to the MVT.** Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then for some \( c \in (a, b) \), \( f(b) - f(a) = f'(c) \cdot (b - a) \). (prove this)

6. In the problem session, you will show that if \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( f(0) = 0 \) and \(|f'(x)| \leq M\), then \(|f(x)| \leq M|x|\). Check that this result holds for \( f(x) = \sin(x) \).

7. (Candice Corvetti) Consider the statement: If \( f : \mathbb{R} \to \mathbb{R} \) is continuous, then the image of every compact set is compact.

(a) Is the statement true or false?

(b) Write the converse of the statement.

(c) For the converse, prove it or give a counterexample.
1. Use the Product Rule, plus induction, to prove that \((x^n)' = n \cdot x^{n-1}\).

2. Suppose that \(f : \mathbb{R} \to \mathbb{R}\) satisfies \(f(0) = 0\) and \(|f'(x)| \leq M\). Prove that \(|f(x)| \leq M|x|\).

The following result is the reason why we need the Mean Value Theorem to do calculus.

3. Corollary 2 to the MVT. On an open interval where \(f'\) is always 0, \(f\) is constant. (prove this)

   Note: Corollary 2 may seem obvious, but it is almost false! Recall the Cantor function, defined in Page 17 # 4. This is a non-constant, continuous function on \([0, 1]\), with derivative 0 everywhere except on the Cantor set, which has measure (total length) 0.

4. (Isaac Kleisel-Murphy) Consider the sequence of functions \(f_n(x) = x^n\) on \((0, 1)\), which converges to \(f(x) = 0\). Does the Proposition in Page 22 # 1 imply that \(f_n \to f\) uniformly, thus giving a counterexample to the Proposition? Explain.
Real Analysis

Cauchy Mean Value Theorem. If \( f, g : [a, b] \to \mathbb{R} \) are continuous on \([a, b]\) and differentiable on \((a, b)\), then for some \( c \in (a, b) \), we have
\[
(f(b) - f(a)) \cdot g'(c) = (g(b) - g(a)) \cdot f'(c).
\]

1. Check the Cauchy MVT for \( f(x) = \sin(\pi x) \) and \( g(x) = x \) on \([0, 1]\). \textit{Hint}: draw a picture

2. Show that the MVT follows directly from the Cauchy MVT (so, the MVT is a corollary of the Cauchy MVT), using a particular choice of \( g(x) \).

3. \textit{Proof of the Cauchy MVT.} First, suppose that \( g(a) = g(b) \). Then we can use (a) Theorem to find \( c \in (a, b) \) such that \( g'(c) = 0 \), which yields the result because (b) ______________________.
   Now suppose \( g(a) \neq g(b) \). Consider the function
   \[
   \phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot (g(x) - g(a)).
   \]
   \( \phi \) is continuous and differentiable because it is a sum and product of continuous and differentiable functions. \( \phi(a) = 0 \) because (c) ______________________, and \( \phi(b) = 0 \) because (d) ______________________. So by (d) Theorem, we can find \( c \in (a, b) \) such that \( \phi'(c) = 0 \), which yields the result because (e) (finish the proof).

Increasing and decreasing functions. Let \( U \subset \mathbb{R} \) be an open set, and let \( f : U \to \mathbb{R} \). We say that \( f \) is:

- \textit{increasing} if, for all \( a, b \in U \), \( a < b \implies f(a) \leq f(b) \).
- \textit{strictly increasing} if, for all \( a, b \in U \), \( a < b \implies f(a) < f(b) \).
- \textit{decreasing} if, for all \( a, b \in U \), \( a < b \implies f(a) \geq f(b) \).
- \textit{strictly decreasing} if, for all \( a, b \in U \), \( a < b \implies f(a) > f(b) \).

In calculus, you used the derivative \( f'(x) \) to tell when \( f(x) \) was increasing or decreasing. This works because of the Mean Value Theorem!

Corollary 3 to the MVT. If \( f'(x) > 0 \) for all \( x \in U \), then \( f \) is strictly increasing on \( U \).

4. \textit{Proof.} Suppose that \( U \subset \mathbb{R} \) is an open set, and \( f : U \to \mathbb{R} \), and \( f'(x) > 0 \) for all \( x \in U \). We will show that, for all \( a, b \in U \), \( a < b \implies f(a) < f(b) \). (prove this) \textit{Hint}: Corollary 1

5. Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Consider the following statement: If \( m \) is any number between \( f'(a) \) and \( f'(b) \), then there exists \( c \in (a, b) \) such that \( m = f'(c) \).
   (a) Explain how this statement is different from that of the MVT. \textit{Hint}: draw a picture
   (b) Prove the statement. \textit{Hint}: There is a detailed hint on p. 109 of \textit{Introduction to Analysis} by Rosenlicht, from which this problem was taken.
1. We showed in Page ___ # ___ that $f$ is differentiable at $x_0$ if and only if it is well-approximated by its tangent line, $L(x) = f(x_0) + f'(x_0)(x - x_0)$. Show that the tangent line $L(x)$ matches the value and the first derivative of $f$ at $x_0$, i.e. $L(x_0) = f(x_0)$ and $L'(x_0) = f'(x_0)$. Also draw a picture to illustrate this.

If $f$ has enough derivatives, we can approximate $f$ with a polynomial that matches not only the value and the first derivative of $f$, but also as many higher-order derivatives as we like, using a polynomial of degree $n$. This is the idea of Taylor’s Theorem:

**Taylor’s Theorem.** Let $U \subset \mathbb{R}$ be an open set, and let $f : U \to \mathbb{R}$ be $n + 1$ times differentiable. Then for all $a, b \in U$, there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$ 

Here, the last term $\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$ is the error term, i.e. the difference between $f(x)$ and its polynomial approximation of degree $n$.

2. Show that the Mean Value Theorem is a corollary of Taylor’s Theorem, with $n = 0$.

3. Use Taylor’s Theorem to show that the function $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \pm \frac{x^n}{n!}$, for $n$ odd, differs from $\sin x$ by at most $\frac{x^{n+1}}{(n+1)!}$ on the interval $[-\pi, \pi]$.

4. Let $g(x) : \mathbb{R} \to \mathbb{R}$ be differentiable at $p \in \mathbb{R}$, and suppose that $g(p) \neq 0$. Show that there exists an interval $I$, containing $p$, such that $g(x) \neq 0$ for all $x \in I$.

**Partitions.** Let $a, b \in \mathbb{R}$ with $a < b$. A partition of $[a, b]$ is given by a (finite) sequence $x_0, x_1, x_2, \ldots, x_N$ such that $a = x_0 < x_1 < \cdots < x_N = b$. The width of the partition is $\max\{x_i - x_{i-1} : i = 1, \ldots, N\}$.

5. If you partition $[a, b]$ using the partition $x_0, x_1, x_2, \ldots, x_N$, how many subintervals of the form $[x_{i-1}, x_i]$ do you get? If they are equally spaced, what is the length of each?

**Riemann sums.** A Riemann sum for $f$, corresponding to the partition $x_0, x_1, x_2, \ldots, x_N$ of $[a, b]$, is $\sum_{i=1}^{N} f(x'_i) \cdot (x_i - x_{i-1})$, where $x'_i \in [x_{i-1}, x_i]$ is a representative point in each subinterval.

6. Compute the Riemann sum for $f(x) = x^2$ on the interval $[1, 4]$, with $N = 4$. Use a partition, and a representative point in each interval, that no one else in the class will think of. Draw a picture, including the rectangles whose areas sum up to the Riemann sum value.

7. Let $U, V \subset \mathbb{R}$ be open sets, and let $f : V \to \mathbb{R}$ and $g : U \to V$. Prove that if $f$ and $g$ are both increasing, then $f \circ g$ is increasing.

8. **Dramatic foreshadowing.** Please draw a picture for each part.

(a) Show that the $x$-axis \{$(x, y) \in \mathbb{R}^2 : y = 0$\} is not compact in $\mathbb{R}^2$.

(b) Show that the circle \{$(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 = 1$\} is compact in $\mathbb{R}^2$. 

April 2018 27 Diana Davis
1. Proof of Taylor’s Theorem. We will show that, if $U \subset \mathbb{R}$ is an open set, and $f : U \rightarrow \mathbb{R}$ is $n+1$ times differentiable, then for all $a, b \in U$, there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b - a)^{n+1}.$$ 

Our goal is to find a number $c \in (a, b)$ to make the above equation true. If $a = b$, the equation holds trivially because (a) $f(b) = f(a)$. Now, $\phi(x) = f(b) - f(a)$, and $f(b)$ has the form $P_n$ such that $\phi(b) = 0$. So we have $c \in (a, b)$ such that $\phi'(c) = 0$.

Now we will assume $a \neq b$. Let $P$ be the real number such that

$$f(b) - \left[ f(a) + \frac{f'(a)}{1!}(b - a) + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n \right] = \frac{P}{(n+1)!}(b - a)^{n+1}. \tag{1}$$

We can find such a $P$ because (b) $f(x)$ is differentiable on $U$. Now, $\phi(x)$ is differentiable on $U$ because it is a sum of differentiable functions: All of the terms of the form $(b - x)^k$ are differentiable because they are polynomials, and the $f', f'', \ldots, f^{(n)}$, are differentiable because (c) $f(x)$ is differentiable. Let $\psi(x) = f(b) - \left[ f(x) + \frac{f'(x)}{1!}(b - x) + \cdots + \frac{f^{(n)}(x)}{n!}(b - x)^n \right] - \frac{P}{(n+1)!}(b - x)^{n+1}.$ \tag{2}

Here the difference between equations (1) and (2) is (d) $\psi(b)$. Now, $\psi(x)$ is differentiable on $U$ because it is a sum of differentiable functions: All of the terms of the form $(b - x)^k$ are differentiable because they are polynomials, and the $f', f'', \ldots, f^{(n)}$, are differentiable because (e) $f(x)$ is differentiable. Let $\gamma(x) = \psi(x) - \psi(b)$.

Now $\gamma(b) = 0$ because (f) and $\gamma(a) = 0$ because (g). So by (h) Theorem, there exists $c \in (a, b)$ such that $\gamma'(c) = 0$.

Now we find $\gamma'(x)$:

$$\gamma'(x) = \frac{d}{dx} \left[ f(b) - \left[ f(x) + \frac{f'(x)}{1!}(b - x) + \cdots + \frac{f^{(n)}(x)}{n!}(b - x)^n \right] - \frac{P}{(n+1)!}(b - x)^{n+1} \right]$$

$$= 0 - \left( f'(x) + f'(x)(-1) + (b - x)f''(x) \right)$$

$$- \left( f''(x) \frac{2(b - x)}{2}(-1) + \frac{(b - x)^2}{2}f'''(x) \right)$$

$$- \cdots$$

$$- \left( f^{(n)}(x) \frac{n(b - x)^{n-1}}{n!} + \frac{(b - x)^n}{n!}f^{(n+1)}(x) \right)$$

$$+ \frac{P}{(n+1)!} (n + 1)(b - x)^n(-1)$$

$$= -\frac{(b - x)^n}{n!}f^{(n+1)}(x) - \frac{P}{(n+1)!} (n + 1)(b - x)^n(-1). \tag{3}$$

Here each line of equation (3) is an application of the (i) , and we can simplify it to equation (4) by (j) . So we have $c \in (a, b)$ such that

$$0 = \gamma'(c) = -\frac{(b - c)^n}{n!}f^{(n+1)}(c) - \frac{P}{(n+1)!} (n + 1)(b - c)^n(-1),$$

which we can solve for $P = f^{(n+1)}(c)$ (k) (show the steps to do this), as desired.
Real Analysis

2. On the graphs of \( f(x) \) defined on \([a, b]\) below, draw in rectangles for the Riemann sum corresponding to each of the following choices for the representative point \( x'_i \) in \([x_{i-1}, x_i]\):

(a) \( f(x'_i) = \max\{ f(x) : x \in [x_{i-1}, x_i] \} \)
(b) \( f(x'_i) = \min\{ f(x) : x \in [x_{i-1}, x_i] \} \)
(c) \( x'_i = x_{i-1} \)
(d) \( x'_i = x_i \)
(e) What are the sums in parts (c) and (d) called?

3. Explain why, as the width of the associated partition decreases, the Riemann sum for \( f \) on \([a, b]\) approaches the area under \( f(x) \) over the interval \([a, b]\).

Integrability. Let \( a, b \in \mathbb{R} \) with \( a < b \), and let \( f : [a, b] \to \mathbb{R} \). Then \( f \) is Riemann integrable on \([a, b]\) if there exists a number \( A \) such that, for all \( \epsilon > 0 \), there exists \( \delta \) such that, if we take any partition of width \( \delta \), and if we take \( S \) to be any Riemann sum associated to such a partition, then \( |A - S| < \epsilon \). In this case, we say that

\[
A = \int_a^b f(x) \, dx
\]

is the Riemann integral of \( f \).

4. Consider the function defined on \([0, 3]\) by: \( f(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \cup [2, 3] \\ 1 & \text{if } x \in (1, 2) \end{cases} \).

(a) Find a partition of \([0, 3]\) of width \( \leq 1/4 \), and draw it on the interval \([0, 3]\).

Find the value of each of the following Riemann sums, when the representative point \( x'_i \):

(b) is the left endpoint of each interval;
(c) yields the maximum value of \( f \) on its interval;
(d) yields the minimum value of \( f \) on its interval.
(e) Find a partition of \([0, 3]\) of width \( \leq 1/10 \), and repeat parts (b)-(d).

5. (Continuation) Refer to the definition of Riemann integrability, given above.

(a) For \( \epsilon = 1/2 \), can you find an \( A \) and a \( \delta \) that satisfy the definition?
(b) For \( \epsilon = 1/10 \), can you find an \( A \) and a \( \delta \) that satisfy the definition?
(c) What are the maximum and minimum values of a Riemann sum associated to a partition of width \( 1/4 \)? And what are the maximum and minimum values for width \( 1/10 \)?
(d) Is \( f(x) \) Riemann integrable?
(e) Explain why, if the difference between the maximum and minimum Riemann sum values for a function \( f \) approaches 0 as the partition width approaches 0, then \( f \) is integrable.
1. Prove, from the definition, that \( f(x) = 3 \) is Riemann integrable on \([0, 1]\).

2. Let \( U, V, W \subset \mathbb{R} \) be open sets, and let \( f : V \to W \) and \( g : U \to V \). Prove that if \( f \) and \( f \circ g \) are both increasing, then \( g \) is increasing. In order to prove this, you will need an additional hypothesis for \( f \) and one for \( g \); determine what these are, and include them in the statement you prove.

3. Prove that \( 1 + 2 + \cdots + n = n(n + 1)/2 \).

4. Prove, using the definition of the Riemann integral, that

\[
\int_{a}^{b} \left( f(x) + g(x) \right) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx.
\]

*Hint:* Such a proof will require \( \epsilon \) (probably \( \epsilon/2 \)), \( \delta \) (probably \( \delta = \min\{\delta_1, \delta_2\} \)), and a chain of inequalities with summations. Use the sum \( \int_{a}^{b}(f + g) \, dx \) as the number \( A \) in the definition.
1. Show that, if \( f : [a, b] \to \mathbb{R} \) is integrable, and \( f(x) \geq 0 \) for all \( x \in [a, b] \), then
\[
\int_{a}^{b} f(x) \, dx \geq 0.
\]

2. Compute directly from the definition that \( \int_{0}^{1} x^2 \, dx = 1/3 \), as follows:
   
   (a) Divide \([0, 1]\) into \( n \) subintervals of width \( 1/n \), and evaluate the following Riemann sum at the right endpoint of each interval, and show that the result is
   \[
   \sum_{k=1}^{n} f(x) \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2.
   \]

   (b) Use the formula \( \sum_{k=1}^{n} k^2 = \frac{n(2n+1)(n+1)}{6} \) and take the limit as \( n \to \infty \).

3. Consider the function defined on \([0, 1]\) by: \( \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \).

   (This function is called the characteristic function of the rationals.)
   
   (a) Find a partition of \([0, 1]\) of width \( \leq 1/4 \), and draw it on the interval \([0, 1]\).
   
   Find the value of each of the following Riemann sums, when the representative point \( x' \):
   
   (b) is the left endpoint of each interval;
   (c) yields the maximum value of \( \chi_{\mathbb{Q}} \) on its interval;
   (d) yields the minimum value of \( \chi_{\mathbb{Q}} \) on its interval.

   (e) Find a partition of \([0, 1]\) of width \( \leq 1/10 \), and repeat parts (b)-(d).

   (f) For \( \epsilon = 1/2 \), can you find an \( A \) and a \( \delta \) that satisfy the definition of Riemann integrability? How about for \( \epsilon = 1/10 \)?

   (g) What are the maximum and minimum values of a Riemann sum associated to a partition of width \( 1/4 \)? How about for width \( 1/10 \)?

   (h) Is it possible to get a value of \( 0.123456789 \) for a Riemann sum of \( \chi_{\mathbb{Q}}(x) \) on \([0, 1]\)? Which possible values between 0 and 1 can you get as a Riemann sum?

   (i) Is \( \chi_{\mathbb{Q}}(x) \) Riemann integrable?

4. You have seen that the set \( \{(-1)^n/n : n \in \mathbb{N}\} \subset \mathbb{R} \) is not compact. Show that, by adding just one point to the set, you can make it compact. This is called one-point compactification. Can you think of any other sets that can be “compactified” in this manner?
5. Consider again the ruler function \( f(x) = \begin{cases} 1/q & \text{if } x = p/q \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \), now over just \([0, 1]\).

(a) For how many values of \( x \) is it true that \( f(x) > 1/4? \)

(b) Explain why, given any \( N > 0 \), there are only finitely many \( x \) for which \( f(x) > 1/N \).

(c) Show that, in fact, given any \( N > 0 \), the number of values of \( x \) with \( f(x) > 1/N \) is bounded by \( N^2/2 \).

(d) What should the value be for \( \int_0^1 f(x) \, dx \)? We will calculate it next time.

6. Theorem. Every continuous, real-valued function is integrable on \([a, b]\).

Proof. We will show that any sequence of Riemann sums whose partition widths \( \to 0 \) is Cauchy. This will prove the result, because the sequence of Riemann sums is a Cauchy sequence of real numbers, which converges because \( (a) \). The limit it converges to is then the Riemann integral \( \int_a^b f(x) \, dx \).

We know that \( f \) is uniformly continuous, because \( (b) \). So, given any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
|x - y| < \delta \implies |f(x) - f(y)| < \epsilon. \tag{1}
\]

Consider two Riemann sums, each with width less than \( \delta/2 \). Their subintervals intersect (if we break up \([a, b]\) at all the places where either of the partitions has a subinterval break) in smaller subintervals of width also less than \( \delta/2 \), because \( (c) \). On each subinterval, the values \( f(x) \) from the two Riemann sums \( S^1 \) and \( S^2 \) come from points at distance at most \( \delta/2 \) from a point in the intersection, and hence at distance at most \( \delta \) from each other, because \( (d) \). By (1), these values differ by at most \( \epsilon \). Let \( x^1_i \) and \( x^2_i \) be the representative values chosen for \( S^1 \) and \( S^2 \), respectively. Summing over the smaller subintervals, we see that the Riemann sums can differ by at most

\[
\left| \sum (f(x^1_i) - f(x^2_i))(x_i - x_{i-1}) \right| \leq \sum |f(x^1_i) - f(x^2_i)| (x_i - x_{i-1}) \tag{2}
\]

\[
\leq \sum \epsilon (x_i - x_{i-1}) \tag{3}
\]

\[
= \epsilon \sum (x_i - x_{i-1}) \tag{4}
\]

\[
= \epsilon (b - a). \tag{5}
\]

Justify the lines \( (e) \) \( (f) \) \( (g) \) \( (h) \). Since \( b - a \) is finite, we can make \( \epsilon (b - a) \) as small as we like, so the sequence is Cauchy.
Real Analysis

1. Is \( f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \) integrable on \([0, 1]?)? Prove your answer correct.

2. Consider the following function \( f: \mathbb{R} \to C \) from points on the \( x \)-axis to points on the circle \( C = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 = 1\} \): For any point \((p, 0)\) on the \( x \)-axis, draw the line segment from \((p, 0)\) to the top point \((0, 2)\) of the circle. Let \( f(p, 0) \) be the point where the line segment intersects the circle. Draw a picture.
   (a) Find \( f(0, 0), f(2, 0), f(-2, 0) \), and \( f \) of one more point of your choice.
   (b) Show that \( f \) is continuous. Is \( f \) a bijection?

Fundamental Theorem of Calculus. Let \( f \) be a continuous function on \([a, b]\).

I. \( \frac{d}{db} \int_a^b f(x) \, dx = f(b) \).

II. If \( f(x) = F'(x) \), then \( \int_a^b f(x) \, dx = F(x) \bigg|_{x=a}^{x=b} = F(b) - F(a) \).

3. Use the FTC to compute \( \int_0^1 x^2 \, dx \). Was this easier or harder than Page 29 # 2?

4. Compute \( \frac{d}{db} \int_a^b x^2 \, dx \), using (a) FTC (I) (b) FTC (II).

Derivatives and integrals with sequences of functions.

5. Let \( f_n(x) = x^n \). Sketch \( f_n(x) \) on \([0, 1]\) for at least five values of \( n \), on one picture.
   (a) Find \( \int_0^1 f_n(x) \, dx \), as a function of \( n \).
   (b) Find the (pointwise) limit function \( f \) of \( f_n \), i.e. \( f(x) = \lim_{n \to \infty} f_n(x) \).
   (c) Does \( f_n \to f \) uniformly?
   (d) Find \( \int_0^1 f(x) \, dx \).
   (e) Say whether you can switch the limit and in the integral in this case, i.e. whether
   \[
   \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \lim_{n \to \infty} f_n(x) \, dx.
   \]

6. Show that, if \( f, g: [a, b] \to \mathbb{R} \) are integrable, and \( f(x) \leq g(x) \) for all \( x \in [a, b] \), then
   \[
   \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.
   \]

7. Let \( f(x) \) be the ruler function. Prove that \( \int_0^1 f(x) \, dx = 0 \).

Hint: Fix \( \epsilon > 0 \), choose \( N \) such that \( 2/N < \epsilon \), and then choose \( \delta < 1/N^3 \). Take a partition of width less than \( \delta \), note that there are \( N^3 \) subintervals in the partition, and note that the number of subintervals contain a point \( x \) with \( f(x) > 1/N \) is bounded by \( N^2 = 2 \cdot N^2/2 \). Then bound the possible Riemann sum values between 0 and \( \epsilon \) to complete the proof.
1. Write the definition(s) for each term:
   (a) continuous  (b) uniformly continuous  (c) inverse image  (d) Cantor set
   (e) Cantor function  (f) bounded function  (g) sequence of functions
   (h) pointwise convergence  (i) uniform convergence  (j) uniformly Cauchy
   (k) differentiable  (l) ruler function  (m) increasing  (n) strictly increasing
   (o) partition  (p) Riemann sum  (q) integrable

2. State the following theorems:
   (a) Rolle’s Theorem
   (b) Mean Value Theorem
   (c) Cauchy Mean Value Theorem
   (d) Taylor’s Theorem

3. For each of the following, say whether it is True or False. For those that are false, give a counterexample.
   (a) A bounded function on a bounded set is uniformly continuous.
   (b) If $f(x)$ is differentiable at $p$, then $f(x)$ is continuous at $p$.
   (c) The continuous image of a compact set is compact.
   (d) If $f_n \to f$, and $|f'_n(x)| \leq 1$, then $f_n \to f$ uniformly.
   (e) For $f(x) = \sin x$, there exists $c \in [0, \pi/2]$ such that $f'(c) = 1/4$.
   (f) Every continuous function is uniformly continuous.

4. For each term in problem 1, write down a result that uses it.

5. Prove that a differentiable function on $[a, b]$ with bounded derivative is uniformly continuous.

6. Let $\mathcal{U}, \mathcal{V}, \mathcal{W} \subset \mathbb{R}$ be open sets, and let $f : \mathcal{V} \to \mathcal{W}$ and $g_n : \mathcal{U} \to \mathcal{V}$ be continuous functions. Prove or give a counterexample:
   (a) If $g_n \to g$, then $f \circ g_n \to f \circ g$.
   (b) If $g_n \to g$ uniformly, then $f \circ g$ is continuous.
   (c) If $g_n \to g$ uniformly, then $g$ is continuous.

7. Write down questions or problems that you wish to discuss in class.
1. Show that, if $|f(x)| \leq M$ for all $x \in [a,b]$, then $\left| \int_a^b f(x) \, dx \right| \leq M(b-a)$.

*Hint:* You can do this from scratch, but it is easier to use a previous result.

**Proof of the Fundamental Theorem of Calculus.**

2. Proof of I. We have

$$\frac{d}{db} \int_a^b f(x) \, dx = \lim_{h \to 0} \frac{\int_a^{b+h} f(x) \, dx - \int_a^b f(x) \, dx}{h} = \int_b^{b+h} f(x) \, dx \quad (1)$$

(a) Justify equation (1). For (2), we use the fact that if $f$ is integrable on $[a,c]$ and $a < b < c$, then $\int_a^c f = \int_a^b f + \int_b^c f$, whose proof is straightforward; we omit it here. Now if $h > 0$, we have

$$\min_{|x-b| \leq |h|} f(x) \leq \frac{\int_b^{b+h} f(x) \, dx}{h} \leq \max_{|x-b| \leq |h|} f(x), \quad (3)$$

by (b) Page __ # __ . (c) Justify why equation (3) also holds when $h < 0$.

(d) Explain why, as $h \to 0$, the left and right sides of (3) both approach $f(b)$.

(e) Finish the proof, that

$$\frac{d}{db} \int_a^b f(x) \, dx = \frac{\int_b^{b+h} f(x) \, dx}{h} = f(b).$$

3. Proof of II. By (1), we have

$$\frac{d}{db} \left( F(b) - \int_a^b f(x) \, dx \right) = F'(b) - f(b) = f(b) - f(b) = 0.$$  

(a) Justify the equalities above. By (b) Page __ # __ , there is a constant $C$ such that

$$\frac{d}{db} \left( F(b) - \int_a^b f(x) \, dx \right) = 0 \implies F(b) - \int_a^b f(x) \, dx = C.$$  

(c) Finish the proof by setting $b = a$ and deducing the desired statement.

4. Let $F(x) = \int_0^x e^{-t^2} \, dt$. Compute $F'(x)$ and $F'(0)$.

5. *One point compactification* We know that the $x$-axis in $\mathbb{R}^2$ is not compact, while a unit circle in $\mathbb{R}^2$ is compact, by Page __ # __ . We also know that the continuous image of a compact set is compact, by Page __ # __ . Show that the set $\mathbb{R} \cup \{\infty\}$ is compact.  

**Terminology:** Here $\{\infty\}$ is known as “the point at infinity.”

6. Let $g_n(x) = x/n$ on $[0, 1]$. Repeat Page 30 # 5 for this function.

7. Write down Rolle’s Theorem for a function $f : [a, b] \to \mathbb{R}$. Then give a counterexample to the theorem when each of the following hypotheses is omitted:

(a) $f$ is differentiable on $(a, b)$  
(b) $f$ is continuous on $(a, b)$  
(c) $f(a) = f(b)$
1. Let \( h_n(x) = \begin{cases} 
0 & \text{if } x = 0 \\
n & \text{if } 0 \leq x \leq 1/n \\
0 & \text{if } 1/n < x < 1 
\end{cases} \) on \([0,1]\). Repeat Page 30 # 5 for this function.

2. Make a conjecture: For a sequence of functions \( f_n \) on \([a,b]\), when can you switch the limit and the integral? We will prove a Theorem about this next time.

Prove the following Corollaries to the Fundamental Theorem of Calculus. Each one is stated in a colloquial way that is easy to remember, and then restated in a precise form that is easier to prove.

3. **Corollary 1.** A continuous function has an antiderivative.

   If \( f \) is a continuous, real-valued function on an open interval \( U \subset \mathbb{R} \), then there exists a real-valued function \( F \) on \( U \) such that \( F'(x) = f(x) \).

4. **Corollary 2.** Antiderivatives differ by a constant.

   If \( F \) and \( G \) are both antiderivatives of \( f \), then \( F - G = C \) for some constant \( C \).

5. **Theorem.** Integrable functions are bounded.

   **Proof.** We will show that, if \( f : [a, b] \rightarrow \mathbb{R} \) is integrable, then \( f \) is bounded on \([a, b]\). **Hint:** One way is to prove it directly, showing that integrability implies that the function value is bounded at every representative point. Another way is to prove the contrapositive, that an unbounded function is not integrable because the Riemann sums do not converge.

6. (Continuation) Prove or find a counterexample to the converse: Bounded functions are integrable.

   **Definition** Given a partition \( x_0, \ldots, x_N \) of a (possibly infinite) interval, a **step function** is a function that is constant on each bin of the partition. Note that the interior intervals in the definition may be closed on the right and open on the left (as written below), or the opposite, or a mix, as long as each point in the interval has an image:

   \[
   f(x) = \begin{cases} 
   a_i & \text{for } x \in [x_{i-1}, x_i), i = 1, \ldots, N-1 \\
   a_i & \text{for } x \in [x_{N-1}, x_N] 
   \end{cases}, \text{ where each } a_i \in \mathbb{R} \text{ is finite.}
   \]

7. Explore the definition:

   (a) Explain why \( f(x) = \begin{cases} 
0 & \text{if } x \in [0,1] \cup [2,3] \\
1 & \text{if } x \in (1,2) 
\end{cases} \) from Page 28 # 4 is a step function.

   (b) Explain the terminology **step function**.

   (c) Is the Cantor function a step function?

   (d) Is the characteristic function of the rationals a step function?

   (e) Make up an example of a step function that no one else will think of, and graph it.

   (f) Define and explain a step function that occurs in everyday life.

8. Prove that step functions are integrable on \([a, b]\).
1. **Theorem.** For a uniformly convergent sequence of continuous functions $f_n$ on $[a,b]$, you can switch the limit and the integral:

$$
\lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dx = \int_{a}^{b} \lim_{n \to \infty} f_n(x) \, dx.
$$

**Proof.** Given $\epsilon > 0$, choose $N$ such that, for all $x \in [a,b]$, $n > N \implies |f_n(x) - f(x)| < \epsilon$. Then

$$
\left| \int_{a}^{b} f_n(x) \, dx - \int_{a}^{b} f(x) \, dx \right| = \left| \int_{a}^{b} (f_n(x) - f(x)) \, dx \right| 
\leq \int_{a}^{b} |f_n(x) - f(x)| \, dx 
< \int_{a}^{b} \epsilon \, dx = \epsilon (b-a).
$$

Justify lines (b) (1), (c) (2), and (d) (3) above. (e) Complete the proof.

2. (Continuation) Give a counterexample to the above Theorem when each of the following hypotheses is omitted: (a) $f_n$ are continuous, (b) $f_n$ converge uniformly.

3. (Continuation) Prove or find a counterexample to the following converse of the above Theorem: If you can switch the limit and the integral for a sequence of functions $f_n$, then the sequence is uniformly convergent to a continuous limit.

When you learn calculus, you learn that “a function is differentiable if its derivative is continuous.” Let’s explore that statement.

4. Let $f(x) = \begin{cases} 
-x^2, & x \leq 0 \\
x^2, & x > 0 
\end{cases}$.

(a) Find $f'(x)$ and $f''(x)$. Graph all three functions.

(c) Is $f(x)$ differentiable? Is $f'(x)$ continuous?

(d) Is $f'(x)$ differentiable? Is $f''(x)$ continuous?

5. The function $g(x)$ in the upcoming Page P12 # 2 is differentiable, but $g'(x)$ is not continuous. Is it a counterexample to the above statement?

6. Recall the Cantor function $f_C(x)$ on $[0,1]$, defined in Page 17 # 4. Find $\int_{0}^{1} f_C(x) \, dx$. *Hint:* draw a picture

**Sequences.** Given a sequence of real numbers $\{x_n\}_{n=1}^{\infty} = \{x_1, x_2, \ldots\}$, we can construct a related sequence, called the sequence of partial sums, $\{S_n\}_{n=1}^{\infty}$, where $S_k = x_1 + x_2 + \cdots + x_k$.

7. For each of the following, give the first five partial sums of the series. Then give a formula for the $n^{th}$ partial sum:

(a) $\{x_n\} = 1, 1, 1, \ldots$ (b) $\{x_n\} = 1/2, 1/4, 1/8, \ldots$ (c) $\{x_n\} = 0.1, 0.01, 0.001, \ldots$
Real Analysis

Prove the following Corollary to the Fundamental Theorem of Calculus:

1. **Corollary 3.** If $F$ is the antiderivative for $f$, then $\int_a^b f(x) \, dx = F(b) - F(a)$.

   *More formally: If $U \subset \mathbb{R}$ is open and $F : U \to \mathbb{R}$ has continuous derivative $f$, then for $a, b \in U$, $\int_a^b f(x) \, dx = F(b) - F(a)$.*

When you learn calculus, you learn that “a function is differentiable if its derivative is continuous.” Let’s continue exploring that statement. In this problem, you may assume that $x^2$ and $\sin(x)$ are continuous and differentiable everywhere, and that $1/x$ is continuous and differentiable when $x \neq 0$.

2. Let $g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$.

   (a) Show that $g(x)$ is continuous on $\mathbb{R}$. (Also graph $g(x)$.)

   (b) Show that $g(x)$ is differentiable, and find $g'(x)$. (Also graph $g'(x)$.)

   (c) Is $g'(x)$ continuous? Prove your answer correct.

3. Recall the Cantor set $C$. Let $\chi_C$ be the characteristic function of the Cantor set: It is 1 on $C$, and 0 otherwise. Compute $\int_0^1 \chi_C \, dx$.

   *Hint:* Given $\epsilon > 0$, choose $n$ such that $(2/3)^n < \epsilon$ and $\delta < 1/3^n$, and use the partition of $[0, 1]$ determined by the intervals of the Cantor set in the $n^{th}$ step of construction.

4. Prove the formula for the sum of a geometric series: If $a, r \in \mathbb{R}$ and $|r| < 1$,

   $$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1 - r}.$$

   *Hint:* telescoping
**Theorem.** Let \( f_n : U \to \mathbb{R} \), where \( U \subset \mathbb{R} \) is open. Suppose that \( f_n' \) is continuous for all \( n \), \( f_n' \) converges uniformly on \( U \), and for some \( a \in U \), \( \{ f_n(a) \} \) converges. Then

(A) \( \lim f_n \) exists,

(B) \( \lim f_n \) is differentiable, and

(C) \( (\lim f_n)' = \lim f_n' \).

1. Rewrite the above Theorem in a “colloquial” manner that is easier to remember.

2. **Proof.** By (a) \( \int_a^x f_n'(t) \, dt = f_n(x) - f_n(a) \). Taking the limit of both sides yields

\[
\lim_{n \to \infty} \left( f_n(x) - f_n(a) \right) = \lim_{n \to \infty} \left( \int_a^x f_n'(t) \, dt \right)
\]

(1)

\[
= \int_a^x \lim_{n \to \infty} f_n'(t) \, dt.
\]

(2)

Here (2) is by (b). For the left-hand side of (1), let \( g(x) \) be the function such that \( \left( f_n(x) - f_n(a) \right) = \int_a^x g(t) \, dt \). Our goal is to show that \( g = f' \).

Let’s do that now. Again by (c) \( \lim_{n \to \infty} \left( f_n(x) - f_n(a) \right) \) exists and equals \( \int_a^x g(t) \, dt \). Since \( \lim_{n \to \infty} f_n(a) \) exists by assumption, \( \lim_{n \to \infty} f_n(x) \) also exists for all \( x \in U \), because (d). This proves (A). Let \( \lim_{n \to \infty} f_n(x) = f(x) \).

Then \( f(x) - f(a) = \int_a^x g(t) \, dt \) for all \( x \in U \), because (e). So by (f) \( f'(x) = g(x) \).

This shows that \( f \) is differentiable, proving (B). Finally, we can rewrite \( f'(x) = g(x) \) as \( (\lim f_n)' = \lim f_n' \) because (f), proving (C).

**Infinite Series.** An *infinite series* is an expression of the form \( x_1 + x_2 + x_3 + \cdots = \sum_{i=1}^{\infty} x_i \).

An infinite series with partial sums \( S_n \) converges to \( S \) (or “has sum \( S \)”) if \( S = \lim_{n \to \infty} S_n \).

In this case, we write \( \sum_{i=1}^{\infty} x_i = S \).

3. For the three infinite series in Page 34 # 7, say whether the series converges, and if so, to what limit. Also express the series and its sum in Sigma notation (\( \sum \)).
Real Analysis

**Geometric series.** A series of the form \( a + ar + ar^2 + ar^3 + \cdots \) is called a geometric series.

4. In the figure, the line has slope 1/3, the squares are inscribed between the line and the \( x \)-axis, and the largest square has area \( a \).
   
   (a) Find the area of the second-largest square.
   
   (b) Find the area of the dark square.
   
   (c) Write an expression for the total area of all of the squares in the picture.
   
   (d) Explain the terminology “geometric series.”

5. Consider the harmonic series \( 1 + 1/2 + 1/3 + 1/4 + \cdots \).
   
   (a) Find the sum of the first 10 terms, and then the first 100 and 1000 terms (use a calculator – don’t type the terms by hand!). Does it seem to be converging to some number? Please answer this last question even if you have already read the next sentence below.
   
   (b) Show that the series diverges to infinity. *Hint:* 
   
   \[
   \sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \geq \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \cdots
   \]

6. Let \( f_n = \begin{cases} 1 & \text{if } x = p/q \in \mathbb{Q} \text{ and } q \leq n \\ 0 & \text{otherwise} \end{cases} \) on \([0, 1]\), where \( p/q \) is in lowest terms as usual.
   
   (a) Plot \( f_n \) for \( n = 1, 2, 3, 4, 5 \) in five different colors on the same axes.
   
   (b) For a given \( n \), is \( f_n \) is integrable on \([0, 1]\)? Why or why not?
   
   (c) Find a function \( f \) so that \( f = \lim f_n \).
   
   (d) Is \( \lim f_n \) integrable? Why or why not?

7. As \( n \to \infty \), which of the following sequences grows the fastest? Put them in order. (The point of this problem is to experiment and guess; we’ll prove the order in a later problem.)

   For a fixed \( k > 1, x > 1 : \quad n^k \quad x^n \quad \log n \quad n! \)

   *Hint:* To get an idea of what is going on, let \( k = 3, x = 2 \), and write out the first 10 terms of each sequence, for \( n = 1, 2, \ldots, 10 \).
Convergence and divergence of sequences. We have already seen that a sequence \( x_n \) converges to \( x \) if \( \lim_{n \to \infty} x_n = x \), and diverges otherwise. Applying the definition of a limit, \( x_n \) converges if there exists \( x \) such that, given any \( \epsilon > 0 \), there exists \( N \) such that \( n > N \implies |x_n - x| < \epsilon \).

1. Show that the sequence \( x_n = 1/2, 3/4, 7/8, 15/16, \ldots \) converges to 1: Given any \( \epsilon > 0 \), show how to construct an \( N \) (that depends on \( \epsilon \)) such that \( n > N \implies |x_n - 1| < \epsilon \).

2. The definition that “a sequence diverges if it does not converge” seems a bit difficult to check in practice. Here is a proposed alternative definition: A sequence \( x_n \) diverges if, for any \( M > 0 \), there exists \( N > 0 \) such that \( n > N \implies |x_n| > M \). What do you think of this definition?

3. Find an expression, consisting of a single term, for the sum \( a + ar + ar^2 + \cdots + ar^n \).

4. (Continuation) Write a true statement: The geometric series \( a + ar + ar^2 + ar^3 + \cdots \) converges when... Then prove it. Hint: Please make your statement cover all possible cases.

5. Lemma. (We use “Lemma” for a result that we are proving because it is one step needed in the proof of a later result. It is like breaking up a big computer program into smaller functions, and then calling them.) Let \( x_1 + x_2 + \cdots \) be an infinite series, and let \( S_1, S_2, \ldots \) be its associated sequence of partial sums. The sequence \( S_n \) converges if and only if, for all \( \epsilon > 0 \), there exists \( N \) such that \( n, m > N \implies |S_n - S_m| < \epsilon \). (Prove this.)

6. Proposition. The infinite series \( x_1 + x_2 + \cdots \) converges if and only if, for any \( \epsilon > 0 \), there exists \( N \) such that, for all \( m > n > N \), \( |x_{n+1} + \cdots + x_m| < \epsilon \). (Prove this.)

Comparison Test. If \( \sum_{n=1}^{\infty} x_n \) and \( \sum_{n=1}^{\infty} y_n \) are infinite series of real numbers such that \( |x_n| \leq y_n \) for all \( n \), and \( \sum_{n=1}^{\infty} y_n \) converges, then \( \sum_{n=1}^{\infty} x_n \) converges.

7. Use the Comparison Test to show that \( 1/3 + 1/5 + 1/9 + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n+1} \) converges.

8. Let \( a_n, b_n \) be two infinite sequences. Explain why, if \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \), it means that \( b_n \) grows much faster than \( a_n \) as \( n \to \infty \).

The notation for this is \( a^n \ll b^n \). The symbol “\( \ll \)” is read “is much much less than,” and its definition is that the ratio of terms goes to 0 as above.

9. Prove or give a counterexample: A limit of integrable functions is integrable.
1. In this problem you will show that, for any fixed $k$, $n^k \ll 2^n$. You may prove this your own way, or you may use the guidance below.

(a) Let $a_n = \frac{n^k}{2^n}$. Show that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1/2$.

(b) Show that there exists $N > 0$ such that, for all $n > N$, $\frac{1}{2}(1 + 1/n)^k < 3/4$.

(c) Show that, for this value of $N$, $n > N \implies a_{n+1} < 3/4 \cdot a_n$.

(d) Show that, for this value of $N$, $n > N \implies a_{n+\ell} < (3/4)^\ell \cdot a_n$.

(e) Prove that $\lim_{n \to \infty} \frac{n^k}{2^n} = 0$.

2. Prove the Comparison Test. Hint: Apply a previous result.

3. Show that, for any fixed $x > 0$, $x^n \ll n!$.

Hint: For a fixed $x$, let $N$ be the smallest integer such that $\frac{1}{2} \geq \frac{x}{N+1}$. Then show that, for $n > N$, $\frac{x^n}{n!}$ is a product of $n$ fractions (write them out), whose product is bounded by $\frac{x^N}{N!} \cdot (1/2)^{n-N}$, and use a previous result to show that this product approaches 0.

Limit Comparison Test. Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be infinite sums with nonnegative terms. Suppose that $\lim_{n \to \infty} \frac{x_n}{y_n} = L$. Then:

(1) If $L$ is a finite number (i.e. $0 < L < \infty$), both series converge or both series diverge.

(2) If $L = 0$ and $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

(3) If $L = \infty$ and $\sum_{n=1}^{\infty} y_n$ diverges, then $\sum_{n=1}^{\infty} x_n$ diverges.

4. Use the Limit Comparison Test to show that:

(a) $\sum_{n=1}^{\infty} \frac{10}{2^n}$ converges, by comparing it to $1/2 + 1/4 + 1/8 + \cdots$.

(b) $0.1 + 0.01 + 0.001 + \cdots$ converges, by comparing it to $1/2 + 1/4 + 1/8 + \cdots$.

(c) $\sum_{n=1}^{\infty} \frac{\log n}{n}$ diverges, by comparing it to the harmonic series.

5. Prove that, if an infinite series $\sum x_i$ converges, then its terms $x_i \to 0$.

6. (Continuation) Prove or give a counterexample to the converse: If the terms of an infinite series $\sum x_i$ go to 0, then the infinite series converges.
7. **Definition.** A sequence \( x_n \) *converges* if there exists \( x \) such that, given any \( \epsilon > 0 \), there exists \( N \) such that \( n > N \implies |x_n - x| < \epsilon \). For each of the following, say whether it is a correct negation, and if so, use it to prove that the sequence 1, 2, 3, \ldots\ does not converge. 

*Hint:* draw a picture

(a) A sequence \( x_n \) *does not converge* if there does not exist \( x \) such that, given any \( \epsilon > 0 \), there exists \( N \) such that \( n > N \implies |x_n - x| < \epsilon \).

(b) A sequence \( x_n \) *does not converge* if there exists \( x \) such that, for all \( \epsilon > 0 \), there exists \( N \) such that \( n > N \implies |x_n - x| > \epsilon \).

(c) A sequence \( x_n \) *does not converge* if, for all \( x \), and for all \( \epsilon > 0 \), there exists \( N \) such that \( n > N \implies |x_n - x| > \epsilon \).

(d) A sequence \( x_n \) *does not converge* if, for all \( x \), and for all \( \epsilon > 0 \), there exists \( N \) such that \( n > N \) and \( |x_n - x| > \epsilon \).

(e) A sequence \( x_n \) *does not converge* if, for all \( x \), there exists \( \epsilon > 0 \), such that there exists \( N \) such that \( n > N \) and \( |x_n - x| > \epsilon \).

(f) Write your own definition: A sequence \( x_n \) *does not converge* if...
4. Show that, for any fixed $k, x > 1$, $n^k \ll x^n$. In other words, show that the result of Page 37 # 1 holds for any $x > 1$, not just $x = 2$.

2. Prove the second part of the Limit Comparison Test:

Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be infinite sums with nonnegative terms.

If $\lim_{n \to \infty} \frac{x_n}{y_n} = 0$ and $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

(The proofs of the first and third parts are similar.)

3. Prove that if $\sum_{n=1}^{\infty} a_n$ converges to $A$, and $\sum_{n=1}^{\infty} b_n$ converges to $B$, then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $A + B$.

1. Let $p(n)$ be any polynomial in the variable $n$, e.g. $3n^4 - n^3 + 5$, and let $x > 1$. Prove that $p(n) \ll x^n$. *Hint:* repeated application of a previous result.
Real Analysis

1. Show that, for any fixed $k$, $\log n \ll n^k$:
   (a) Argue that it suffices to show that $\log x \ll x^k$ (i.e. that we need not use whole numbers).
   (b) Let $y = \log x$, so $e^y = x$. Explain how to rewrite $\frac{\log x}{x^k} = \frac{y}{(e^k)^y}$, and argue that this fraction approaches 0 as $x \to \infty$.

2. Rearrangements.
   (a) Compute partial sums of the following series (maybe write them below):
   $$\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \cdots$$
   (b) Compute partial sums of the following series:
   $$\frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - \frac{1}{3} - \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \cdots$$
   (c) Does the sequence in part (a) converge? Does the sequence in part (b)? Explain.

Absolute and conditional convergence. The series $\sum_{n=0}^{\infty} x_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |x_n|$ is convergent. A series that converges, but does not converge absolutely, is conditionally convergent.

3. For each of the following, decide whether it is absolutely convergent, conditionally convergent, or divergent. (Consider doing part (d) before answering the question.)
   (a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$
   (b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$
   (c) $2 \ln 2 - 3/\ln 3 + 4/\ln 4 - 5/\ln 5 + \cdots$
   (d) Using a number line where $-1$ is all the way on the left side of your page and $1$ is all the way on the right, plot the partial sums of the two sequences above, for $n = 1, 2, \ldots, 10$. **Guess** whether each series is convergent, and if so, **guess** to what limit. Can you prove any of your guesses correct with the tests that we have so far?

4. Show that an absolutely convergent sequence is convergent. **Hint:** use a previous result.
Real Analysis

Ratio Test (technical version).

If \( \sum_{n=0}^{\infty} x_n \) is an infinite series where each term is a nonzero real number, then:

1. If there exist \( \rho, N \) such that \[ \left| \frac{a_{n+1}}{a_n} \right| \leq \rho < 1 \] for \( n > N \), the series converges absolutely.

2. If there exists \( N \) such that \[ \left| \frac{a_{n+1}}{a_n} \right| \geq 1 \] for \( n > N \), the series diverges.

5. Using the Ratio Test, say whether the following series converge absolutely, diverge, or whether the test is inconclusive.
   
   (a) \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \)  
   (b) \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \)

6. Prove the Ratio Test, part 2. Hint: use a previous result

7. Show how to use one-point compactification to “compactify” \( \mathbb{R}^2 \). Then watch the 1-minute YouTube video called “Stereographic projection,” by Henry Segerman. (I mean it. You won’t be disappointed.)
1. Proposition. If a series converges absolutely, then all of its rearrangements converge to the same limit. (Whew!)

Proof. Let $\sum a_n$ denote the original series, which converges absolutely to some limit $L$, and let $\sum b_n$ denote its rearrangement. We wish to show that, given any $\epsilon > 0$, there exists $N$ such that $n > N \implies (a)$

Given $\epsilon > 0$, choose $N_1$ such that, if $N, M > N_1$, we have $\left| \sum_{n=1}^{N} a_n - L \right| < \epsilon/2$ and we also have $\sum_{n=N}^{M} |a_n| = |a_M + a_{M+1} + \cdots + a_N| < \epsilon/2$. We can do the first because (b), and the second because (c).

Now let's consider the rearrangement $b_n$. Choose $N_2$ large enough that the first $N_2$ terms of $\sum b_n$ include the first $N_1$ terms of $\sum a_n$. [For example, if $N_1 = 6$ for the first series in Page 38 # 2, we would have (d)]. Choose $N$ so that $N > N_2$. We can do this because (e). Now choose $N_3$ such that the first $N_3$ terms of $\sum a_n$ include the first $N$ terms of $\sum b_n$. [For example, if $N = 7$ for the second series in Page 38 # 2, we would have (f)].

Then $\left| \sum_{n=1}^{N} b_n - L \right| \leq \left| \sum_{n=1}^{N} b_n - \sum_{n=1}^{N_3} a_n \right| + \left| \sum_{n=N_3}^{N} a_n - L \right|$ by (g). Now, we know that the last term on the right is at most $\epsilon/2$, because (h). For the middle term, there are two pieces. All of the terms of the first sum (of $b_n$s) are included in the second sum (of $a_n$s), because (i). The first $N_1$ terms of the second sum are included in the first sum, because (j). So now, the middle term is at most $\epsilon/2$, because (k). (l) (finish the proof)

2. Assume $p > 0$. Show that $\int_{1}^{\infty} \frac{1}{x^p} \, dx = \lim_{R \to \infty} \int_{1}^{R} \frac{1}{x^p} \, dx$ is finite if and only if $p > 1$.

3. Define the sequence of functions $s_n(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \pm \frac{x^{2n+1}}{(2n+1)!}$.

(a) Write out $s_n(x)$ for $n = 1, 2, 3, 4$.

(b) If $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$, write a formula for the function $f_k(x)$.

(c) Use a computer to graph $s_n(x)$ for $n = 0, 1, 2, 3, 4$, and sketch the pictures in your notebook. Do you recognize this family of functions?
Real Analysis

**p-series.** A p-series is a series of the form \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) for \( p > 0 \).

4. On the same picture, graph \( f(x) = 1/x^2 \) for \( x \in (0,10] \), and plot the points \( (n,g(n)) \) for \( g(n) = 1/n^2 \) and \( n = 1, \ldots, 10 \). Explain how to see the p-series for \( p = 2 \) in this picture.

5. Consider the following functions:
   - \( f_p(x) = 1/x^p \),
   - the step function defined by \( U_p(x) = f_p(i) \) if \( x \in [i,i+1) \), where \( i \in \mathbb{N} \),
   - the step function defined by \( L_p(x) = f_p(i+1) \) if \( x \in [i,i+1) \), where \( i \in \mathbb{N} \).

   (a) Make a large, clear graph in the first quadrant of \( L_p(x) \), \( f_p(x) \) and \( U_p(x) \) on the same picture.
   (b) Compute \( \int_1^{\infty} U_p(x) \, dx - \int_1^{\infty} L_p(x) \, dx \), and justify your answer geometrically.

6. (Continuation) Explain why, for any \( R > 1 \) and any \( p > 0 \),
   \[
   \int_1^{R} L_p(x) \, dx < \int_1^{R} f_p(x) \, dx < \int_1^{R} U_p(x) \, dx.
   \]

The **Ratio Test** says:

If \( \sum_{n=0}^{\infty} x_n \) is an infinite series where each term is a nonzero real number, then:

1. If there exist \( \rho, N \) such that \( \left| \frac{a_{n+1}}{a_n} \right| \leq \rho < 1 \) for \( n > N \), the series converges absolutely.
2. If there exists \( N \) such that \( \left| \frac{a_{n+1}}{a_n} \right| \geq 1 \) for \( n > N \), the series diverges.

7. **Proof of the Ratio Test, part 1** (we already proved part 2).
   (a) Explain why the terms before \( N \) make no difference with respect to the convergence or divergence of the series. So for convenience, we may assume \( N = 1 \).
   (b) Use the fact that \( |a_{n+1}| \leq \rho |a_n| \) for all \( n \), to show that \( |a_{n+1}| \leq \rho^n |a_1| \).
   (c) Use the Comparison Test to complete the proof.
1. **Proposition.** If a series converges conditionally, then its terms may be rearranged to converge to any limit, or to diverge to $\pm\infty$, or to diverge by oscillation.

In all of the rearrangements that follow, we’ll keep the positive terms in the same order, and the negative terms in the same order. The idea is that we’ll “front load” positive terms to get big positive limits, and we’ll “front load” negative terms to get big negative limits.

(a) Explain (probably by contradiction) why the sum of all of the positive terms must be $+\infty$, and the sum of the negative terms must be $-\infty$.

(b) Show that the terms of the series go to 0.

(c) Given some desired limit $L$, first take positive terms until you pass $L$ heading right, and then take negative terms until you pass $L$ heading left, and repeat this over and over. Explain why this rearrangement converges to $L$.

(d) Make a rearrangement as follows: Take positive terms until you pass 1, then take one negative term; then take positive terms until you pass 2, then take one negative term, etc. Explain why this rearrangement diverges to $+\infty$.

(e) Explain how to make a rearrangement that diverges to $-\infty$.

We’ll prove divergence by oscillation in the problem session.

**Series of functions.** Let $\{f_k(x)\}$ be a sequence of real-valued functions. The sequence of partial sums is $\{s_n(x)\}$, where $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$.

**Convergent series of functions.** The notation $\sum_{k=1}^{\infty} f_k(x)$ and $\{s_n(x)\}$ mean the same thing. These are called infinite series. The series $\sum_{k=1}^{\infty} f_k(x)$ converges at $p$ if $\sum_{k=1}^{\infty} f_k(p)$ converges, i.e. if $\{s_n(p)\}$ is a convergent series of real numbers.

The series $\sum_{k=1}^{\infty} f_k(x)$ converges on $E$ if it converges for all $p \in E$. If so, then there exists a function $s$ on $E$ such that $s(p) = \lim_{n \to \infty} s_n(p) = \sum_{k=1}^{\infty} f_k(p)$ for each $p \in E$, i.e. $s = \sum_{k=1}^{\infty} f_k$.

2. **Continuity of series of functions.**

(a) Show that, if each $f_k(x)$ is continuous, then $s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$ is continuous for each $n$.

(b) Under what conditions is it true that $f_k(x)$ continuous $\implies s = \sum_{k=1}^{\infty} f_k$ continuous?
3. The equation for the bell curve, or normal distribution, is \( f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \). It models the proportion of population that is \(|x|\) standard deviations from the mean of 0 (the \(1/2\) in the exponent stretches it horizontally to make this happen). So that it measures probabilities, the area under the curve from \(-\infty\) to \(\infty\) is 1 (the \(1/\sqrt{2\pi}\) makes this happen).

(a) For the SAT exam, for each part, the average score is 500 and the standard deviation is 100. Explain why the proportion of the population between 650 and 750 is \( \int_{1.5}^{2.5} f(x) \, dx \).

(b) Integrating this function seems useful. Find an antiderivative for \( f(x) \). (trick question)

(c) One way to integrate it is by switching to a double integral in polar coordinates. Watch my YouTube video “Multivariable calculus, class 24: change of variables” from 5:30 to 19:00, or watch/read any other resource on “integrating the Gaussian distribution” or similar, and write down this calculation.

A downside of the polar coordinates method is that we can only use it to integrate from \(-\infty\) to \(\infty\). In (a) we wanted to integrate from 1.5 to 2.5. So, we need another way. We’ll find one next time.

**Alternating series.** If \( a_n \geq 0 \) for all \( n \), then \( \sum_{n=1}^{\infty} (-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots \) is called an alternating series.

4. **Proposition.** If \( \{a_n\} \) is a decreasing sequence of positive numbers converging to 0, then \( a_1 - a_2 + a_3 - a_4 + \cdots \) converges, to a number less than \( a_1 \).

You may prove this your own way, or you may use the guidance below (answer part (c) in either case):

(a) Consider the sequence of partial sums \( S_n \). Find a way to express \( S_n \) as a sum of nonnegative numbers (you may wish to consider even and odd \( n \) separately), and show that \( \lim S_n \geq 0 \).

(b) Show that \( \lim S_n \leq a_1 \).

(c) Explain why the combined result \( 0 \leq \lim S_n \leq a_1 \) does not prove that \( S_n \) converges.

(d) Show that \( S_n \) converges by showing that \( |S_m - S_n| \leq a_{n+1} \).

5. Show that, for fixed \( x, k > 1 \),

(a) \( x^n \ll n! \), and

(b) \( \log n \ll n^k \ll x^n \ll n! \).
1. Prove that if a series converges conditionally, then its terms may be rearranged to diverge by oscillation.

2. \((p\text{-series test})\) Prove that, for \(p > 0\), \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) converges if and only if \(p > 1\).

   \textit{Hint:} use a previous result.

3. Prove that the series \(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\) converges for all values of \(x\).

4. Consider the metric space consisting of the integers \(\mathbb{Z}\), with the discrete metric: \(d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{if } m \neq n. \end{cases}\)

   Show that this metric space is closed and bounded, but not compact.
1. In this problem, we’ll continue our quest to integrate under the bell curve. Taylor polynomials and term-by-term integration to the rescue!

(a) Explain why \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \) is continuous.

(b) Show that \( f'(0), f''(0), f^{(3)}(0) \) exist, and find their values.

(c) Let \( \{g_k(x)\} = \left\{ \frac{f^{(k)}(0)}{k!} x^k \right\} \). Write out \( g_0(x), g_1(x), g_2(x) \) and \( g_3(x) \).

(d) Let \( \{s_n(x)\} \) be the sequence of partial sums of \( g_k(x) \). Write out \( s_0(x), s_1(x), s_2(x) \) and \( s_3(x) \).

(e) Explain why these are the Taylor polynomials for \( f \), i.e.

\[
s_n(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \cdots + \frac{f^{(n)}(0)}{n!}(x-0)^n.
\]

(f) For which values of \( x \) does the sequence of partial sums converge? In other words, for which values of \( x \) does \( \lim_{n \to \infty} s_n(x) \) exist? What is this limit?

(g) For which values of \( x \) is it true that \( \sum_{k=0}^{\infty} g_k(x) = f(x) \)?

(h) The next Theorem (in its Corollary), tells us that, for such values of \( x \), you can integrate \( \sum g_k(x) \) term by term. Do so, to find a power series expression for \( \int f(x) \).

2. Find a sequence of functions \( f_k(x) \) such that each \( f_k \) is continuous, but \( s = \sum f_k \) is not continuous.
3. Theorem (Integrating series of functions). For a uniformly convergent sequence of continuous functions on \([a, b]\), you can switch the sum and the integral.

Proof. We’ll show that, if \(\{s_n(x)\}\) is a uniformly convergent sequence of continuous real-valued functions on \([a, b]\), then

\[
\sum_{k=1}^{\infty} \int_{a}^{b} f_k(x) \, dx = \int_{a}^{b} \sum_{k=1}^{\infty} f_k(x) \, dx.
\]

(a) Use the Theorem in Page 34 # 1 to prove that

\[
\int_{a}^{b} \left( \lim_{n \to \infty} s_n(x) \right) \, dx = \lim_{n \to \infty} \left( \int_{a}^{b} s_n(x) \, dx \right).
\]

Since \(\{s_n\}\) is uniformly convergent, it converges to some function \(f = \sum f_k(x)\). So

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} \left( \lim_{n \to \infty} s_n(x) \right) \, dx = \lim_{n \to \infty} \left( \int_{a}^{b} s_n(x) \, dx \right) = \lim_{n \to \infty} \int_{a}^{b} \sum_{k=1}^{n} f_k(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} \int_{a}^{b} f_k(x) \, dx = \sum_{k=1}^{\infty} \int_{a}^{b} f_k(x) \, dx,
\]

as desired. Justify equalities (b) (1), (c) (2), (d) (3), (e) (4), and (f) (5).

4. Corollary. For a uniformly convergent series, you can integrate term by term.

(Prove this.)

5. In Page 38 # 5b we guessed that \(1 - 1/2 + 1/3 - 1/4 + \cdots\) converges. Now prove it.

Ratio Test (easier-to-use version). For the series \(\sum_{n=0}^{\infty} x_n\):

1. If \(\lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = \rho < 1\), it converges absolutely.
2. If \(\lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = \rho > 1\), it diverges.

6. Show that this Ratio Test is a corollary of the technical version of the Ratio Test.
1. Theorem (Differentiating series of functions). For a sequence of functions that converges somewhere, whose derivatives are continuous, and whose sequence of partial sums is uniformly convergent, you can switch the derivative and the integral.

Proof. We’ll show that, if \( f_k' \) is continuous for all \( k \), and \( \sum_{k=1}^{n} f_k' \) is uniformly convergent, and \( \sum_{k=1}^{n} f_k(a) \) converges for some \( a \), then \( \lim_{n \to \infty} s_n' = \left( \sum_{k=1}^{n} f_k \right)' = f' \).

(a) Use Page 35 # 1-2 to show that, for a sequence of partial sums \( s_n \), if \( s_n' \) is continuous, and \( s_n' \) is uniformly convergent, and \{\( s_n(a)\)\} converges for some \( a \), then \( \left( \lim_{n \to \infty} s_n \right)' = \lim_{n \to \infty} s_n' \).

Now we have

\[
\lim_{n \to \infty} s_n' = \lim_{n \to \infty} (f_1 + f_2 + \cdots + f_n)' = \lim_{n \to \infty} (f_1' + f_2' + \cdots + f_n') = \lim_{n \to \infty} \sum_{k=1}^{n} f_k' = \sum_{k=1}^{\infty} f_k' = \left( \lim_{n \to \infty} s_n \right)' = \left( \sum_{k=1}^{n} f_k \right)' = f',
\]

as desired. Justify equalities \( b \) (1), \( c \) (2), \( d \) (3), \( e \) (4), \( f \) (5), \( g \) (6).

2. Corollary. For a uniformly convergent series, you can differentiate term by term.

(Prove this.)
Real Analysis

3. Recall that for a partial sum of a geometric series, we have

\[ a + ar + ar^2 + \cdots + ar^n = \frac{a(1 - r^{n+1})}{1 - r}. \]

(a) Rewrite both sides of the above equation using \( a = 1 \) and \( r = -x \).

(b) We showed that a geometric series converges when \( |r| < 1 \). Assuming \( |x| < 1 \), take the limit of both sides of your equation from (a). Then explain why the sequence of partial sums converges, and say what it converges to.

(c) Show that, for any \( k < 1 \), \( f(x) = 1 - x + x^2 - x^3 + \cdots \) converges uniformly on \([-k, k]\).

(d) Rewrite your answer from (b) in the form \( f(x) = \sum_{k=0}^{\infty} f_k(x) \).

(e) Under what conditions is it true that \( \int_a^b f(x) = \sum_{k=0}^{\infty} \int_a^b f_k(x) \, dx \)?

(f) Find an infinite series equal to \( \ln(1 + x) \). Hint: Switch to the variable \( t \) to make things clearer, and integrate \( f(t) \) on \([0, x]\) for \( 0 < x < 1 \).

(g) It turns out that this infinite series function is continuous at \( x = 1 \). Plug in \( x = 1 \) to find the value of \( 1 - 1/2 + 1/3 - 1/4 + \cdots \), resolving our conjecture from Page 38 # 3c.

4. Prove that \([0,1) \subset \mathbb{R}\) and \([0,1) \times [0,1) \subset \mathbb{R}^2\) have the same cardinality, by constructing a bijection (a one-to-one correspondence) from the interval to the square.

5. Prove that, for any set \( X \), the power set of \( X \) (that is, the set of subsets of \( X \)) is strictly larger.

\( \text{Hint:} \) Suppose that there exists a bijection, and show that this leads to a contradiction. The argument is demonstrated in Gallery of the Infinite, by Richard Schwartz, about 3/4 of the way through the book, just after the discussion of the Cantor Set. Here is a link (please do not share it): http://www.math.brown.edu/~res/infinity.pdf

6. We will use induction to prove that all bears are the same color.

\( \text{Base case:} \) Consider a set consisting of one bear. Clearly, all of the bears in the set are the same color.

\( \text{Induction hypothesis:} \) Suppose that it is true that, for sets of bears up to size \( n \), all of the bears in the set are the same color. We will show that the same is true for a set of size \( n + 1 \). Now consider a set of \( n + 1 \) bears (technically called a sleuth of \( n + 1 \) bears). Removing one bear from the set yields \( n \) bears, which by hypothesis are all the same color. Returning that bear and removing a different bear also yields \( n \) bears, which by hypothesis are all the same color. So all \( n + 1 \) bears are the same color, as desired.