Arrays: Vectors and Matrices

Vectors

Vectors are an efficient notational method for representing lists of numbers. They are equivalent to the arrays in the programming language "C".

A typical vector might represent the high temperature every day for a week. Such a vector would have seven elements and could be written as a row vector (a single row),

$$\mathbf{HighTemp} = \begin{bmatrix} 25 & 32 & 33 & 38 & 43 & 45 & 41 \end{bmatrix}$$

or as a column vector (a single column),

$$\mathbf{HighTemp} = \begin{bmatrix} 25 \\ 32 \\ 33 \\ 38 \\ 43 \\ 45 \\ 41 \end{bmatrix}$$

Note, that I will use bold letters when referring to the entire vector (or matrix). In both cases above, the **HighTemp** vector has seven elements, representing Sunday through Saturday. To access the individual elements in the array we use an index. For example, the temperature Sunday would be accessed as HighTemp(1), and is equal to 25. Likewise HighTemp(4)=38 is Wednesday's high temperature.

Matrices

A matrix (singular of matrices) is for our purposes a series of numbers listed in two dimensions. As an example, consider high temperatures collected over a 28 day period (4 weeks). We could write the matrix as a single list 28 elements long, or as a collection of numbers (a matrix) that has 4 rows and 7 columns.

$$\mathbf{HighTemp} = \begin{bmatrix} 25 & 32 & 33 & 38 & 43 & 45 & 41 \\ 42 & 43 & 45 & 46 & 48 & 41 & 39 \\ 39 & 41 & 43 & 47 & 48 & 48 & 47 \\ 50 & 49 & 45 & 48 & 50 & 51 & 53 \end{bmatrix}$$

Now we need two indices to represent the numbers, one for the row, and one for the column. For example the high temperature on the 3rd day of the 2nd week is HighTemp(2,3) and is equal to 45. Note that the index for the row comes first.

We could also write the matrix with rows and columns interchanged. This is referred to as taking the "transpose" of the matrix.

$$\mathbf{HighTemp} = \begin{bmatrix} 25 & 42 & 39 & 50 \\ 32 & 43 & 41 & 49 \\ 33 & 45 & 43 & 45 \\ 38 & 46 & 47 & 48 \\ 43 & 48 & 48 & 50 \\ 45 & 41 & 48 & 51 \\ 41 & 39 & 47 & 53 \end{bmatrix}$$

Note that a vector is the special case of a matrix, where there is only one row or column. Also note that the two vectors given are transposes of each other.

In general a matrix consisting of m·n elements can be arranged in m rows and n columns, yielding an mxn (read m by n) matrix, which we'll call **A**.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The symbol a_{ij} represents the number in the ith row and the jth column.

Matrix operations
For the ensuing discussion assume we have three matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

Equality
Two matrices are equal if they are the same size, and corresponding elements are equal. For example, **A=B**, but **A≠C**.

Addition
Two matrices can be added if they are the same size. Their sum is given by a third matrix whose elements are the sum of the corresponding elements being added.of the two arrays If **D=A+C** then

$$\mathbf{A} + \mathbf{C} = \mathbf{D} = \begin{bmatrix} 1 & 7 \\ 3 & 5 \end{bmatrix}$$

Note that **A+B=B+A**.

Operations with a scalar
A matrix can be multiplied by a scalar (a scalar is a single number) by multiplying each element of the array by that number. The same can be done with addition, subtraction of division with a scalar. For example:

$$2 \cdot \mathbf{A} = \mathbf{D} = \begin{bmatrix} 2 & 8 \\ 4 & 6 \end{bmatrix}$$

To convert the temperatures from the first example to Celsius:
CelcTemp=(**HighTemp**-32)*5/9

Multiplication of two vectors

A row vector can be multiplied by a column vector, in that order, to yield a scalar if and only if the have the same number of elements. If

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then

$$\mathbf{uv} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n, \quad \text{or } \mathbf{uv} = \sum_{i=1}^{n} u_i v_i$$

For example:

$$\begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 = 44$$

.

Multiplication of two matrices

Two matrices, $\mathbf{A}$ and $\mathbf{C}$ can be multiplied together in the order $\mathbf{AC}$ if and only if the number of columns in $\mathbf{A}$ equals the number of rows in $\mathbf{C}$ (in other words, the inner dimensions are equal). If $\mathbf{D} = \mathbf{AC}$, then d_{ij} is the element obtained by multiplying the row vector represented by the ith row of $\mathbf{A}$ by the column vector represented by the jth column of $\mathbf{C}$. For arrays $\mathbf{A}$ and $\mathbf{C}$ with n columns and rows, respectively, we get:

$$d_{ij} = \sum_{k=1}^{n} a_{ik} c_{kj}$$

For example, if D=AC, using the matrices given previously

Note that in general AC≠CA. In this case

$$\mathbf{D} = \mathbf{AC} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 4 \cdot 1 & 1 \cdot 3 + 4 \cdot 2 \\ 2 \cdot 0 + 3 \cdot 1 & 2 \cdot 3 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ 3 & 12 \end{bmatrix}$$

$$\mathbf{E} = \mathbf{CA} = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 5 & 10 \end{bmatrix} \neq \mathbf{D}$$

ÇÈ

Important matrices

Identity Matrix

There is a special matrix called the identity matrix. This is a square matrix has one's along the main diagonal, and 0's elsewhere. Shown are 2x2 and 4x4 identity matrices.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The identity matrix has the special property that for a square matrix $\mathbf{A}$, $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

Inverse Matrix

If the matrix $\mathbf{A}$ has an inverse $\mathbf{G}$, we write $\mathbf{G} = \mathbf{A}^{-1}$, and $\mathbf{GA} = \mathbf{AG} = \mathbf{I}$. Note that some matrices don't have inverses.

Review - Vectors and Matrices

For the following problems let:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{2}{3} & \dfrac{1}{3} \\ -\dfrac{1}{3} & \dfrac{1}{3} \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad V = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

1. What is a_{12}?

2. What is a_{21}?

3. What is $A+B$?

4. What is UV?

5. What is AB?

6. What is AI?

7. What is IA?

8. What is VU?

9. What is BA?

10. How could you use matrix multiplication to find the average of each row of the matrix A?

11. How could you use matrix multiplication to find the average of each column of the matrix A?

Answers - Vectors and Matrices

For the following problems let:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \dfrac{2}{3} & \dfrac{1}{3} \\ -\dfrac{1}{3} & \dfrac{1}{3} \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

1. What is a_{12}? -1

2. What is a_{21}? 1

3. What is **A**+**B**? $\begin{bmatrix} 1\dfrac{2}{3} & -\dfrac{2}{3} \\ \dfrac{2}{3} & 2\dfrac{1}{3} \end{bmatrix}$

4. What is **UV**? 11

5. What is **AB**? $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

6. What is **AI**? $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

7. What is **IA**? (Note, in this case **IA**=**AI**, multiplication isn't generally commutative). $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

8. What is **VU**? (Note, in this case **UV**≠**VU**, because multiplication isn't generally commutative).

$\begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$

9. What is **BA**? $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

10. How could you use matrix multiplication to find the average of each row of the matrix **A**?

$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix}$

11. How could you use matrix multiplication to find the average of each column of the matrix **A**?

$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix}$